

A perspective on energy orthogonality in wave-vortex flow decompositions

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Energy orthogonality

1. Flow decomposition

A flow decomposition is always linear in the sense that you are separating the flow into two or more parts that must add to the total,

$$\begin{bmatrix} p(x, y, z, t) \\ u(x, y, z, t) \\ v(x, y, z, t) \\ w(x, y, z, t) \\ \eta(x, y, z, t) \end{bmatrix} = \begin{bmatrix} p_1 \\ u_1 \\ v_1 \\ w_1 \\ \eta_1 \end{bmatrix} + \begin{bmatrix} p_2 \\ u_2 \\ v_2 \\ w_2 \\ \eta_2 \end{bmatrix}$$

$$\Psi_{\text{tot}} = \Psi_1 + \Psi_2$$

There are infinite possibilities here.

2. Energy orthogonality

Energy to lowest order is quadratic

$$\mathcal{E}[(u, v, w, \eta)] = \frac{1}{2} \int (u^2 + v^2 + w^2 + N^2 \eta^2) dV$$

To now say that energy fluxed **from** somewhere **to** somewhere else, you need energy orthogonality

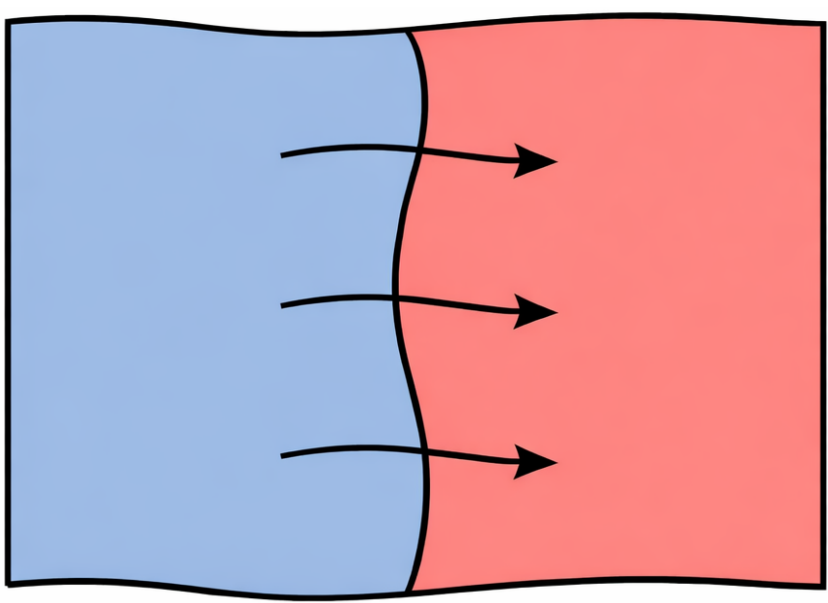
$$\mathcal{E}[(u, v, w, \eta)] = \mathcal{E}[(u_1, v_1, w_1, \eta_1)] + \mathcal{E}[(u_2, v_2, w_2, \eta_2)]$$

$$\Psi_{\text{tot}}^2 = \Psi_1^2 + \Psi_2^2$$

with no quadratic cross term!

3. Spatial energy fluxes

In the spatial domain we decompose the domain into orthogonal (non-overlapping) volumes and measure fluxes from one volume to another,



and now we have a spatial flux.

4. Spectral energy fluxes

In the spectral domain energy fluxes from one *mode* to another mode.

$$\Psi_1^2 \rightarrow \Psi_2^2 \rightarrow \Psi_3^2$$

This is an energy cascade.

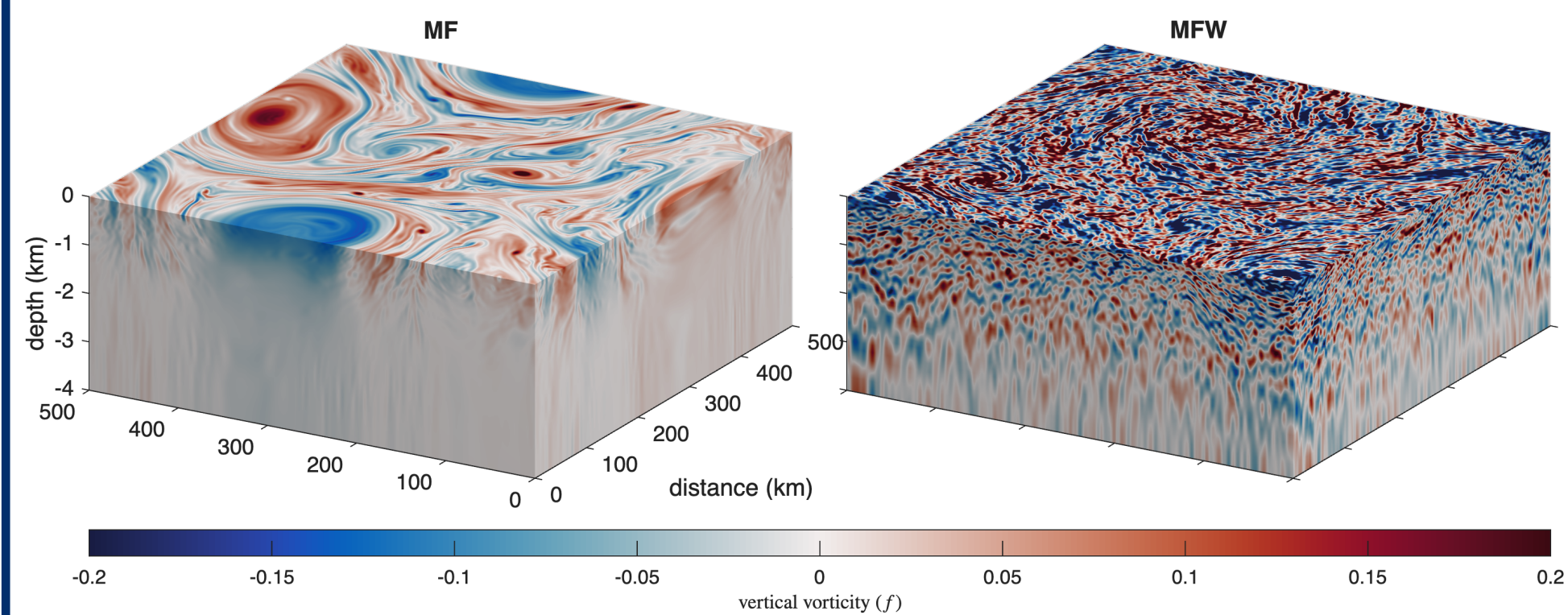
The same works for potential enstrophy.

5. Known orthogonal systems

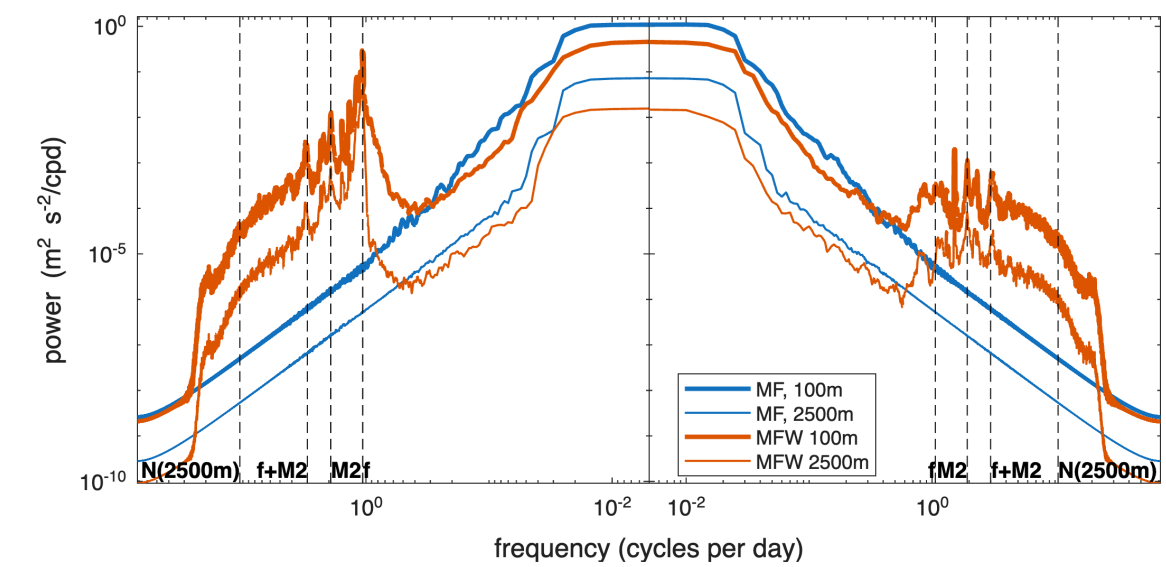
Inertial ranges and cascades *require* an energetically orthogonal basis. Many systems have been solved.

Geometry	Rotating?	Stratified?	Non-hydrostatic?	References
triply-periodic	no	no	yes	Kolmogorov (1941), Onsager (1949)
two-layer	yes	no	no	Veronis (1956)
quasigeostrophy	yes	variable	no	Charney (1971)
triply-periodic	no	constant	yes	Riley et al. (1981), Lelong & Riley (1991)
triply-periodic	yes	constant	yes	Lelong (1989), Bartello (1995)
shallow-water	yes	no	no	Rommel & Smith (2009)
quasigeostrophy w/ $b(0) \neq 0$	yes	variable	no	Smith & Vanneste (2013), Yassin (2021)
free-surface	yes	variable	no	Olbers (1986), Kelly (2016)
spherical shallow-water	yes	no	no	Zagar (2020)
spherical	yes	variable	no	Vasylykevych & Zagar (2021)
doubly-periodic	yes	variable	yes	Early et al. (2021), Early et al. (2026)

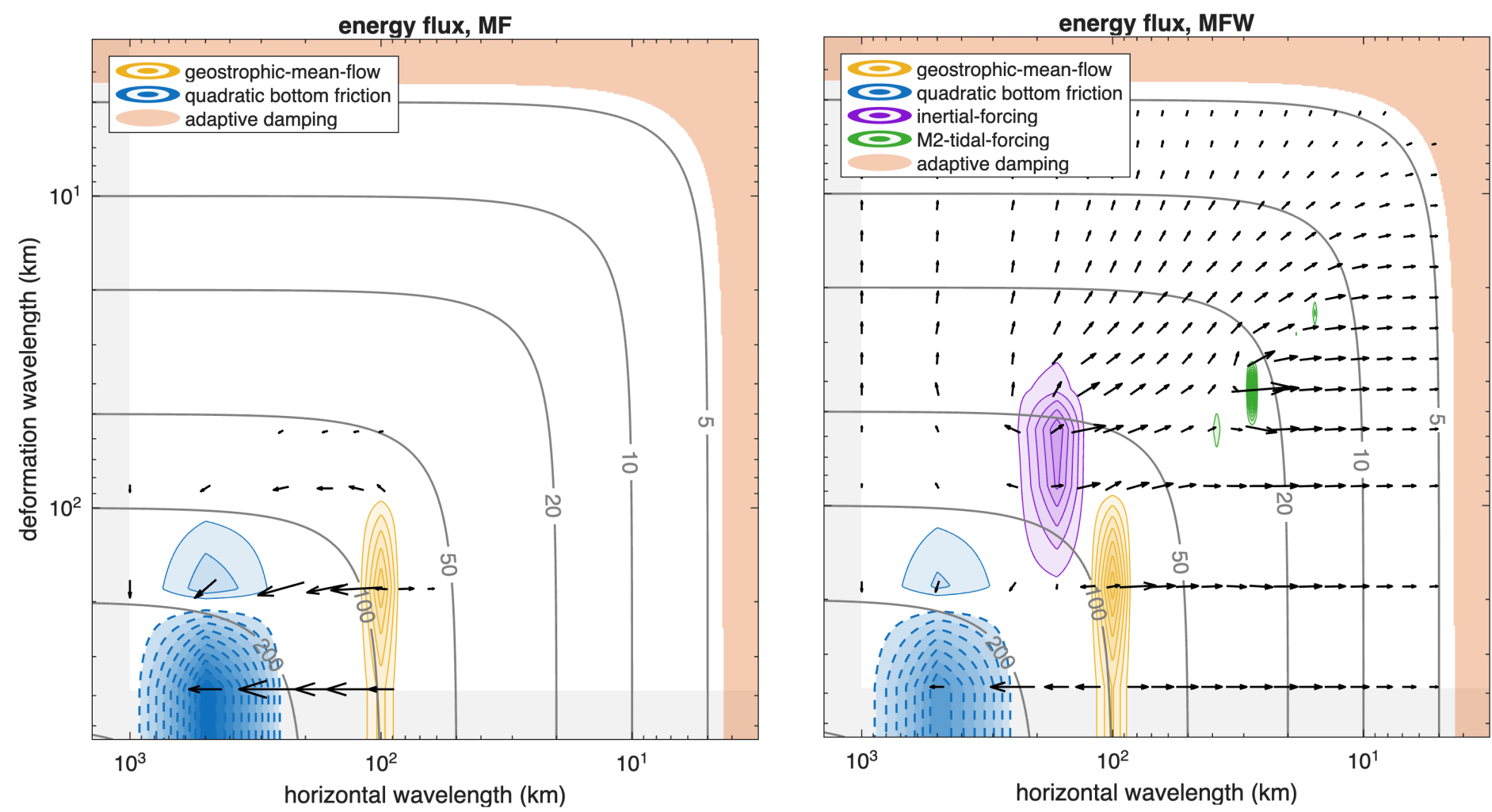
Two Nonhydrostatic Simulations



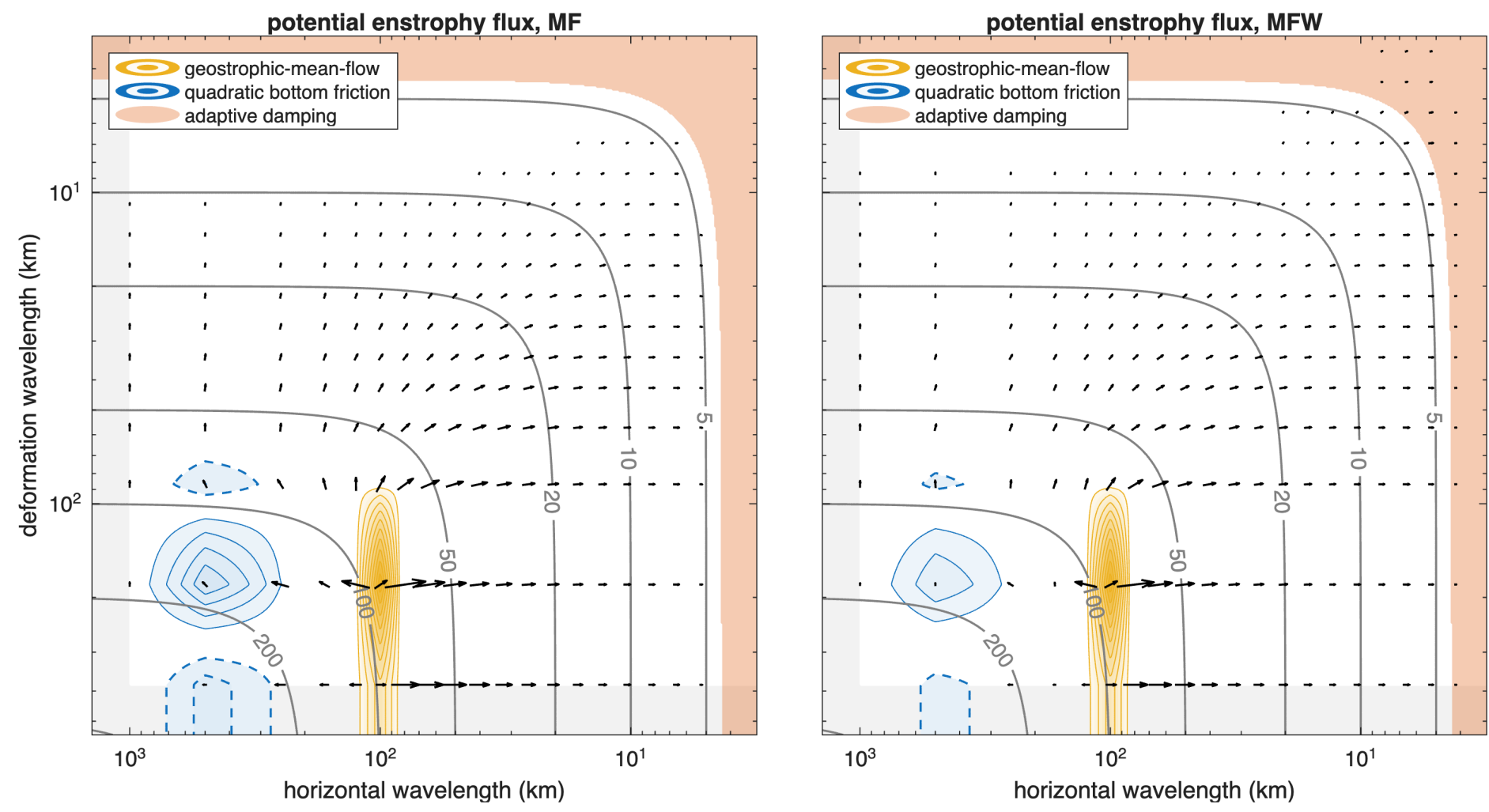
- 500 km x 5000 km x 4000m
- $N^2(z) = N_0^2 \exp(2z/b)$
- Quadratic bottom friction, small-scale damping, diffusivity
- **MF**: Mean Flow geostrophic forcing
- **MFW**: Mean Flow + Wave forcing (tidal & inertial)



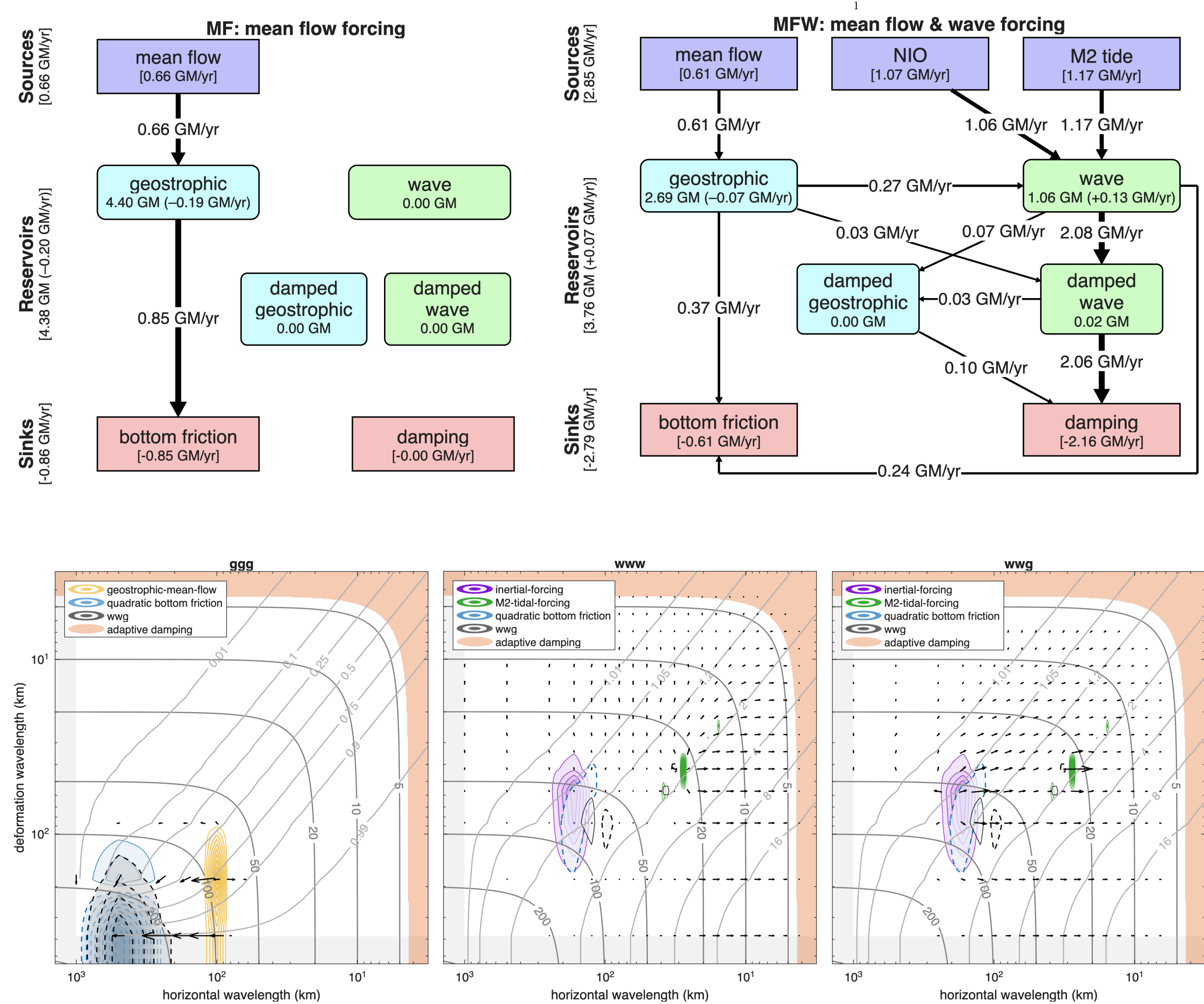
- Exact energy flux spectra



- Exact potential enstrophy flux spectra



Wave & Geostrophic Energy Fluxes

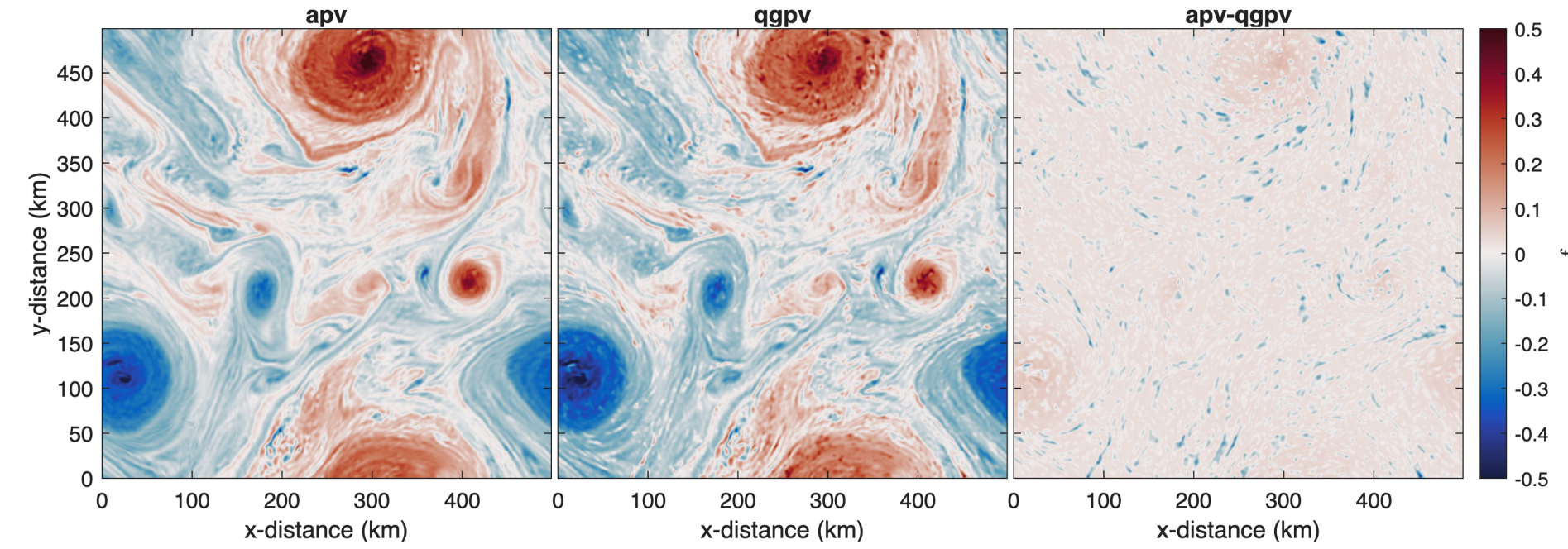


- MF simulation looks like QG
- Wave forcing decreases geostrophic energy
- Geostrophic **inverse cascade**
- Geostrophic transfer to waves
- Wave **forward cascade**

- “ggg” triad behaves like QG in the MFW simulation
- “ggw” triad is essentially zero.
- “wwg” fluxes from forcing
- “www” fluxes to smaller pseudo-wavelength

Error Assessment

- The decomposition is a PV-inversion, of QGPV
- So we can compare APV (exact quantity) with QGPV (approximation)



- Vortex stretching error dominates at large scales
- relative vorticity error dominates at small scales

One iteration of optimal balance, Masur & Oliver, (2020) reduces the error by an order of magnitude, but at the cost of energy orthogonality.

Products & Acknowledgements

Measuring fluxes between wave and geostrophic features in rotating non-hydrostatic flows with variable stratification (2026)

wavevortexmodel.org
Non-hydrostatic model with automatic wave-vortex decomposition

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