

Motivation

Our goal is to study particle dispersion from an internal wave field in a numerical model with conditions matching those observed at Latmix 2011 site 1.

Challenges

- A numerical model requires *exact* internal wave solutions for a bounded domain, whereas observations are necessary *local* in the water column. Can these be reconciled?
- Initializing a numerical model with an internal wave spectrum requires solving $N_x \times N_y$ eigenvalue problems to compute the vertical modes. Can this be done in a fast and accurate way?

Internal Modes

The two most practical vertical eigenvalue problems (EVPs) are and

$$\partial_{zz} G_j = -\frac{N^2 - \omega^2}{gh_j} G_j \quad (\omega\text{-constant EVP})$$

where G is the vertical mode for density/

$$\partial_{zz} G_j - K^2 G_j = -\frac{N^2 - f_0^2}{gh_j} G_j \quad (K\text{-constant EVP})$$

vertical velocity and N is the stratification profile.

Orthogonality & Completeness

- Only modes with the same frequency or same wavenumber are orthogonal to one another.
- If $\omega > N(z)$ anywhere in the water column, the EVP modes will not be a complete basis.

Numerics

- The K-constant EVP must be solved for all resolved wavenumbers in a model.
- EVPs algorithms scale as the *cube* of the grid points in the vertical.

- Standard second order finite differencing produces **garbage** for all but the lowest modes.

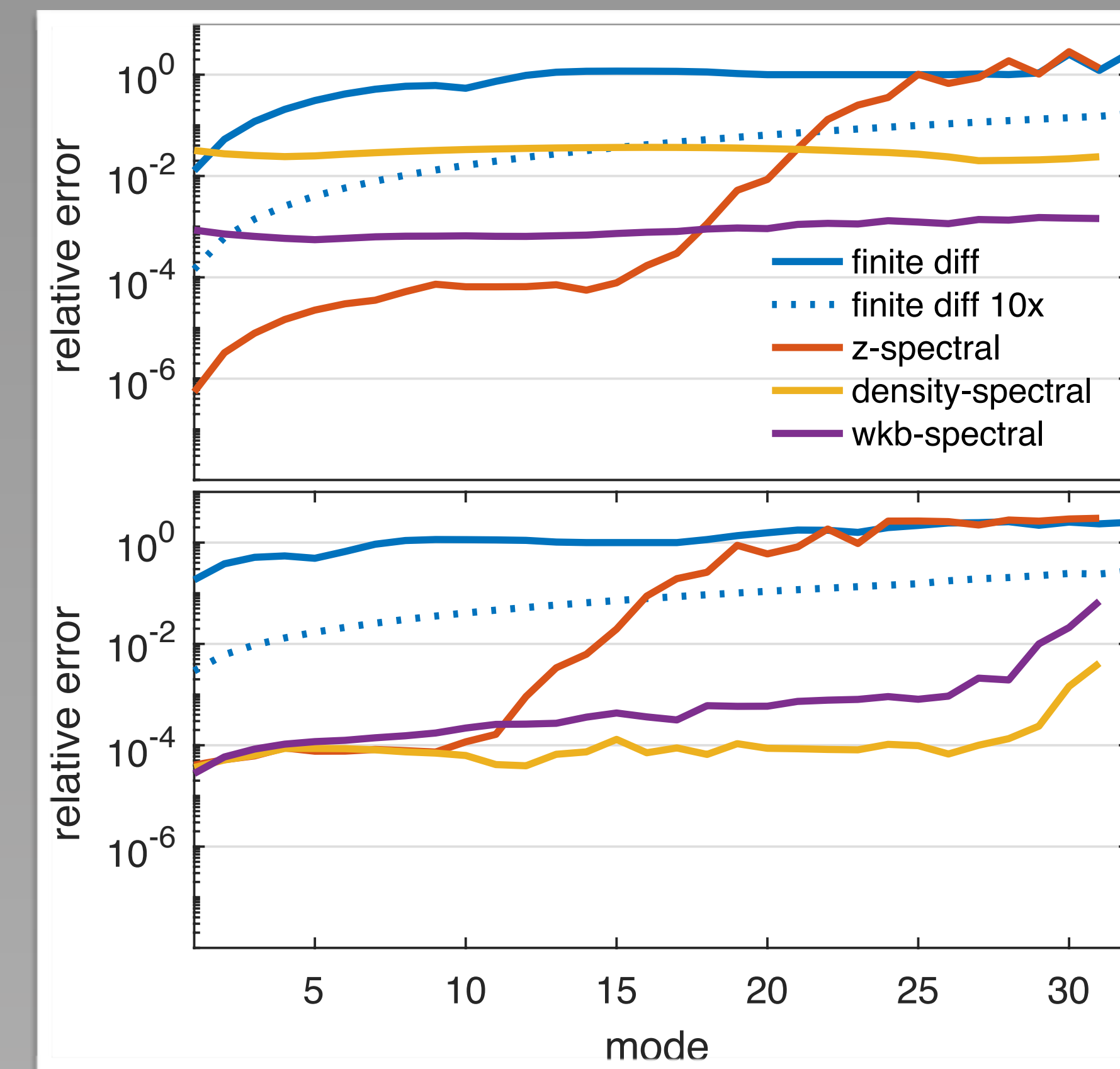


Figure 1: relative error as a function of vertical mode number using 64 evenly spaced grid points in exponential stratification $N(z) = e^{z/b}$ where $b=1300$ m in a 5000 m deep ocean for $K = 0$ (top) and $K = 2\pi/500$ m (Bottom).

Solution

We solve the EVP on a WKB-stretched coordinate using Chebyshev polynomials, which produces very accurate wave modes at small computational cost.

Matlab

A complete Matlab implementation is available, just search for 'GLOceanKit'.
<https://github.com/JeffreyEarly/GLOceanKit/>

Initialize with a density profile and request modes at a frequency or wavenumber:

```
im = InternalModes(rho,z,zOut,latitude);
[F,G,h,omega] = im.ModesAtWavenumber(k);
[F,G,h,k] = im.ModesAtFrequency(omega);
```

The classes will also

- solve using spectral or finite difference methods of arbitrary order,
- solve for the SQG modes,
- solve with an adaptive method near turning points.

Garrett-Munk Spectrum

The GM spectrum is an empirical description of the energy distribution in the internal wave field as a function of vertical mode and frequency,

$$\mathcal{E}_{GM}^H / \rho = E_0 \int_{f_0}^{N_{\max}} B(\omega) d\omega \sum_{j=1}^{\infty} H(j) \text{ where}$$

$$B(\omega) = \frac{f}{\omega} \frac{B_0}{\sqrt{\omega^2 - f^2}}, \quad H(j) = \frac{H_0}{(j^2 + j_*^2)^p}.$$

which implies the *exact* variance relations

$$\langle u^2 + v^2 \rangle(\omega, z) = E_0 B(\omega) \left(1 + \frac{f_0^2}{\omega^2}\right) \Phi(\omega, z)$$

$$\langle \zeta^2 \rangle(\omega, z) = E_0 B(\omega) \left(1 - \frac{f_0^2}{\omega^2}\right) \Gamma(\omega, z)$$

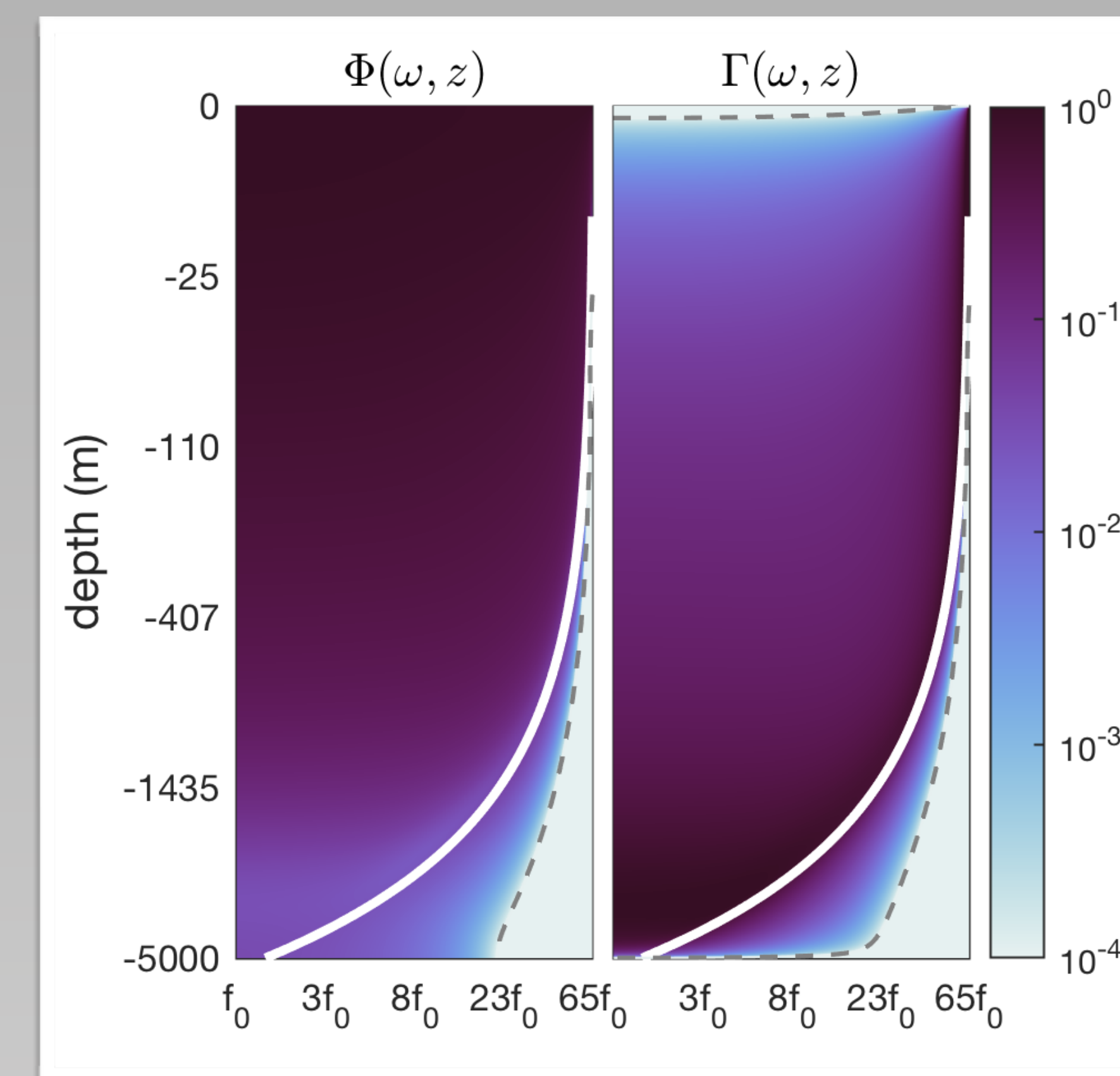
$$\langle w^2 \rangle(\omega, z) = E_0 B(\omega) (\omega^2 - f_0^2) \Gamma(\omega, z)$$

where

$$\Phi(\omega, z) \equiv \sum_{j=1}^{\infty} \frac{1}{h_j} H(j) F_j^2(\omega, z)$$

$$\Gamma(\omega, z) \equiv \frac{1}{g} \sum_{j=1}^{\infty} H(j) G_j^2(\omega, z).$$

are the **vertical structure functions**.



Vertical structure functions for exponential stratification. The color axis is logarithmic to highlight regions of high and low variances. The white line marks the buoyancy frequency, while the dashed gray line marks where the variance drops below 10^{-4} the max.

GM Approximation

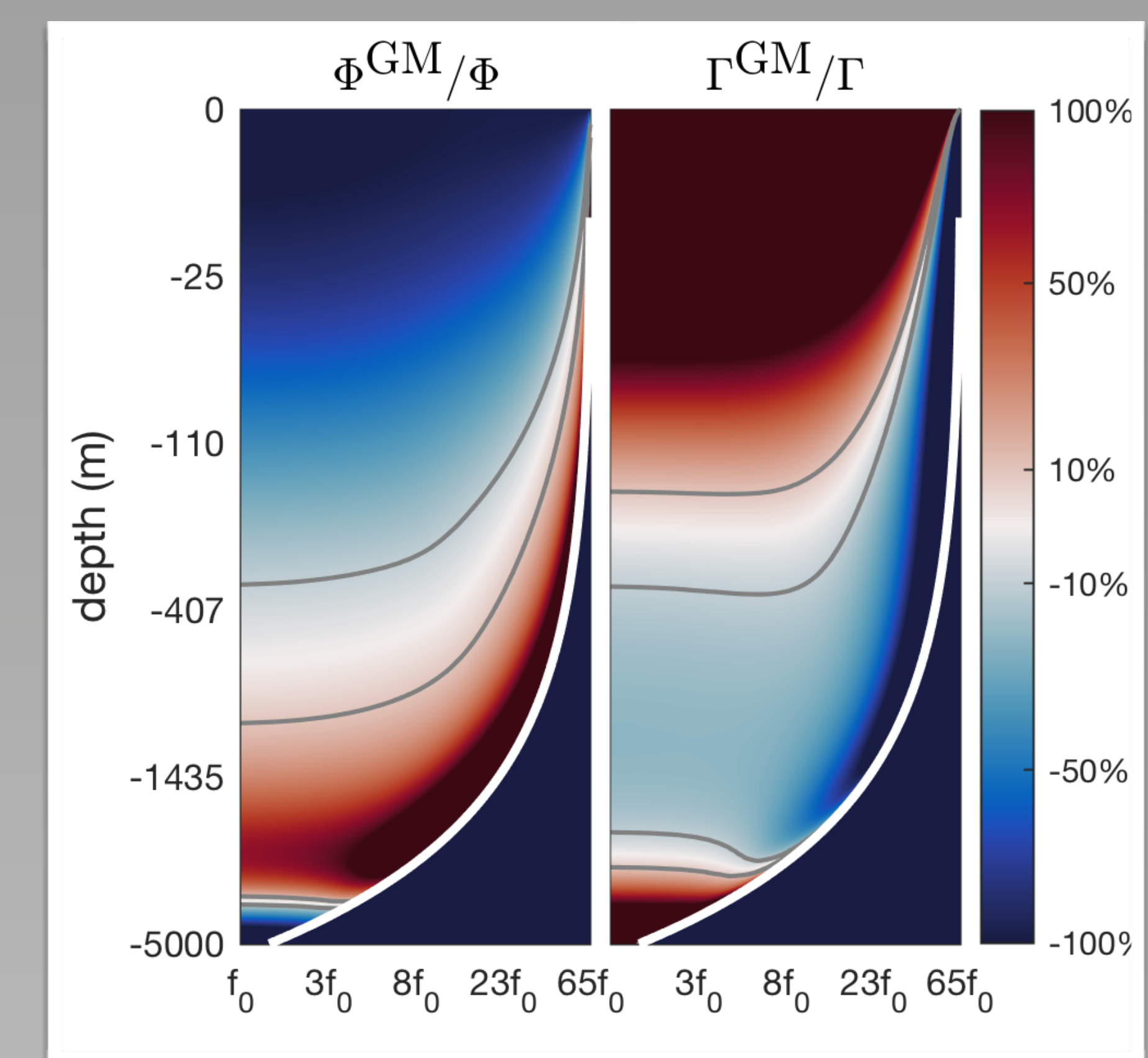
The usual approximations are

- WKB (good!)
- hydrostatic (terrible!)
- boundary free, exponential-like (okay!)

$$\Phi^{GM}(z) = \frac{N(z)}{bN_0}$$

$$\Gamma^{GM}(z) = \frac{1}{bN_0 N(z)}$$

How good is this approximation?



Ratio of the GM approximated structure functions to the exact structure functions. The color bar is scaled such that red and blue indicate regions where the GM approximation overestimates and underestimates variance, respectively. The two regions with less than 10% error are contoured with gray. The buoyancy frequency, $N(z)$, is drawn in white.

Conclusions

- Approximated and true frequency spectra often differ by more than a factor of 2, **but** this can be corrected for any known stratification.
- Vertical wavenumber spectra do not need to be corrected (not shown).
- Challenge complete.

Publications

- "Fast and Accurate Computation of Vertical Modes using Chebyshev Polynomials" submitted to Ocean Modelling
- "Variance Relations of the Garrett-Munk Spectrum" in prep for Journal of Marine Research.