

A TWO-LAYER MODEL FOR INERTIAL OSCILLATIONS IN A SLOWLY VARYING BACKGROUND FLOW

JEFFREY J. EARLY

INCLUDING MANY (MOST) OTHERS AT THIS
MEETING

HOW THIS FITS TOGETHER

1. *A stochastic model* to fit the observations.
2. *A kinematic model* to interpret the parameters.
3. *A dynamical model* that explains the parameter values.

THE STOCHASTIC MODEL

- The stochastic model is chosen to capture the *observed* variability of the Lagrangian drifters.
- Importantly, the model uses as *few* as parameters as possible, to capture as *much* of the structure as possible.

A KINEMATIC MODEL

- The kinematic model tells us how to interpret the statistical model and its parameters as fundamental physical concepts.
- For example, the diffusivity, background slope, oscillation frequency, damping parameter all correspond to kinematic models for particle motion.

A DYNAMICAL MODEL

- Finally, we need a *dynamical* model to explain the particle values of the parameters.
 - Can we explain the observed turbulent background parameters with QG, SQG or frontal dynamics?
 - What physical processes contribute the damping parameter?
 - How about changes in oscillation frequency?

ONE PATH FORWARD...

- Given that,
 1. we trust the mathematicians' statistical model describes the observations well,
 2. we are starting to get a handle on the kinematic interpretation of the model parameters,
- **what is the simplest dynamical model of the ocean that can reproduce all the kinematic elements of the stochastic model?**

WHY SIMPLE?

- Using a simply dynamical model allows us to more easily diagnose how different physical processes control different parameters in the stochastic model.
- Conversely, if we find that the simple dynamical model does not reproduce an observed parameter, but a more robust dynamical model does (e.g., Danioux), it becomes much easier to find the missing physics.

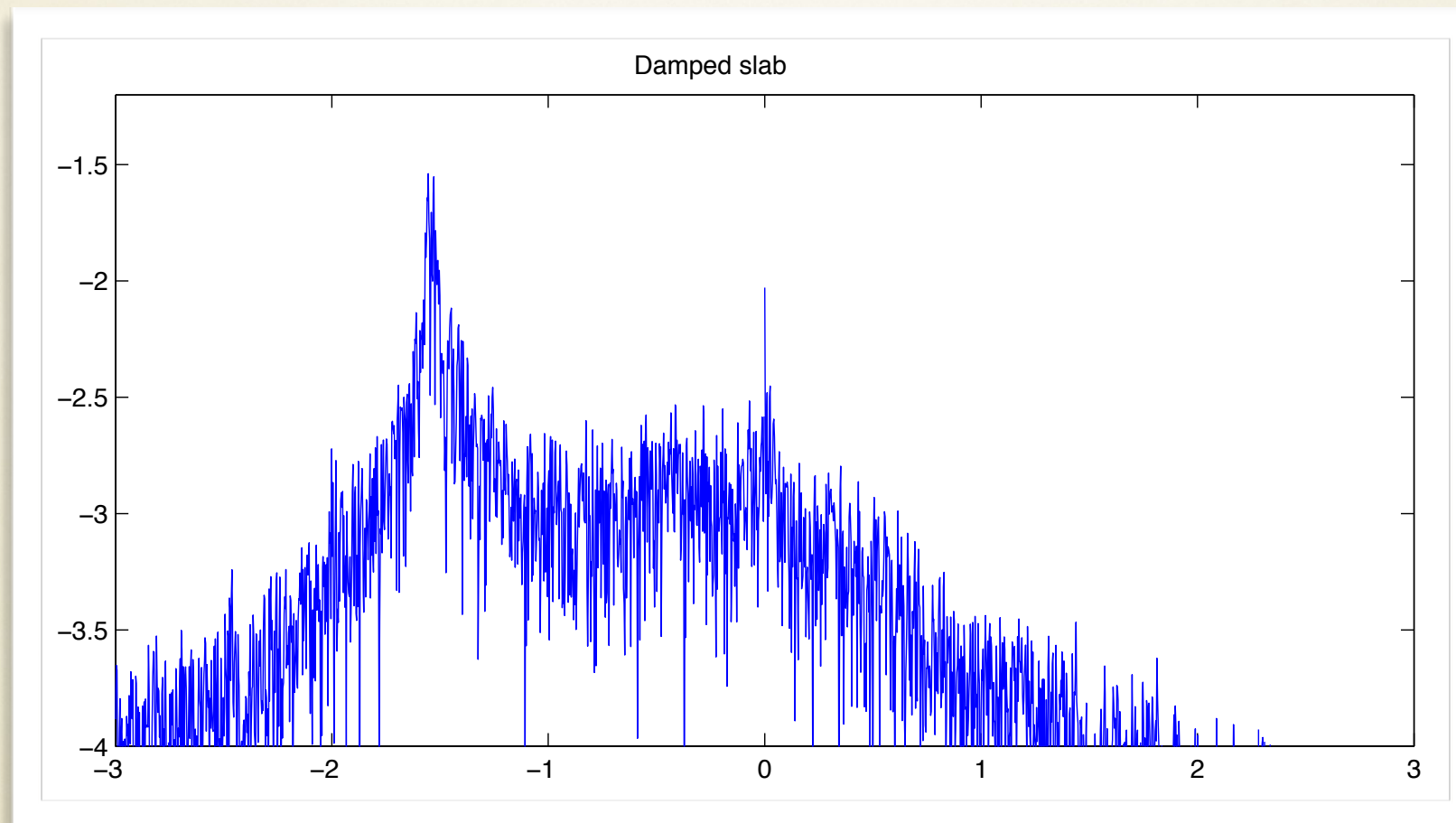
WHY SIMPLE?

- “It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience.” – Einstein

WHAT DYNAMICS?

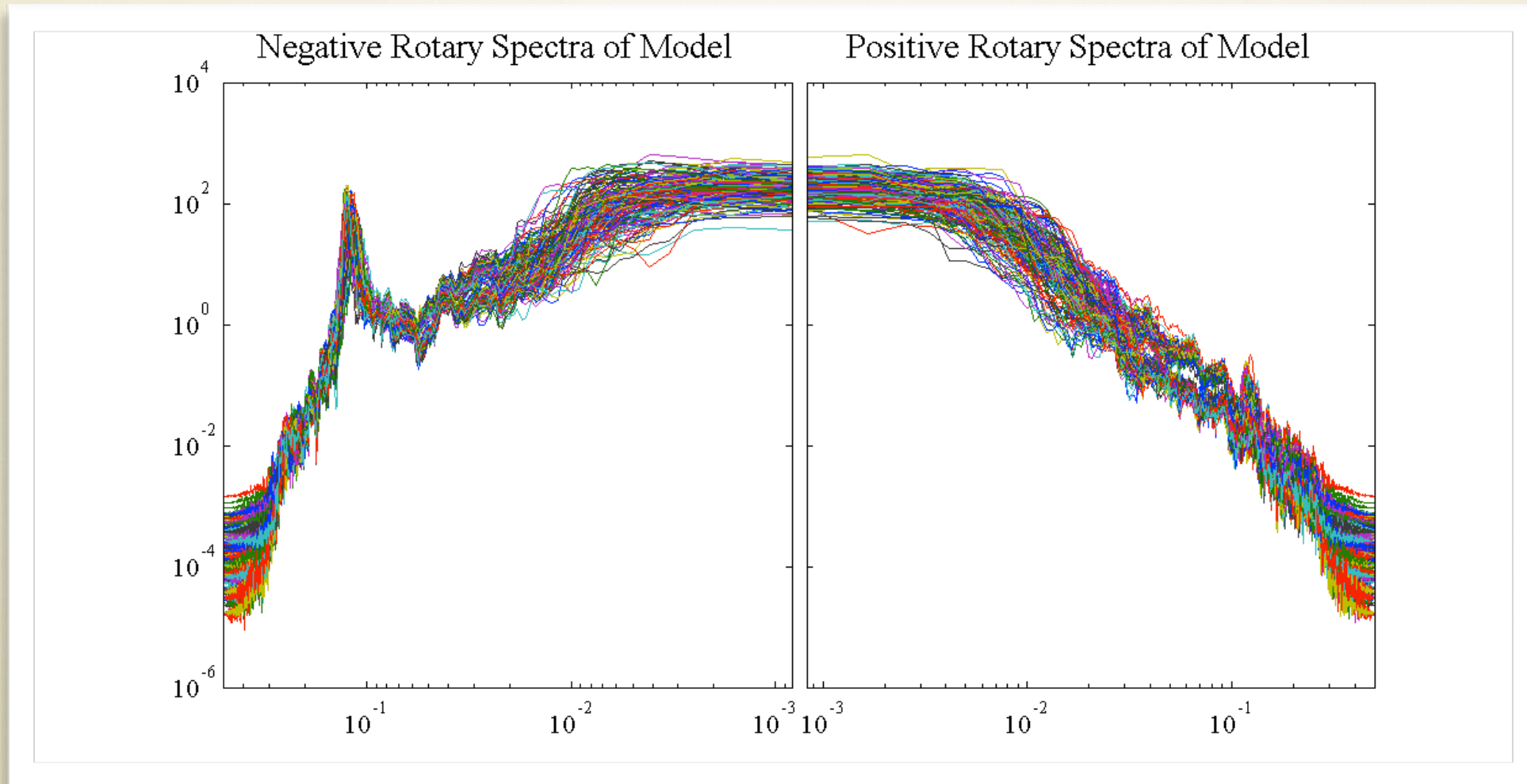
- We suspect our model should include low-frequency turbulence. Maybe something like QG or SQG will match observed parameters?
- It's also pretty clear our model should include inertial oscillations—and a way to extract energy from those oscillations.

INGREDIENT 1: INERTIAL OSCILLATIONS



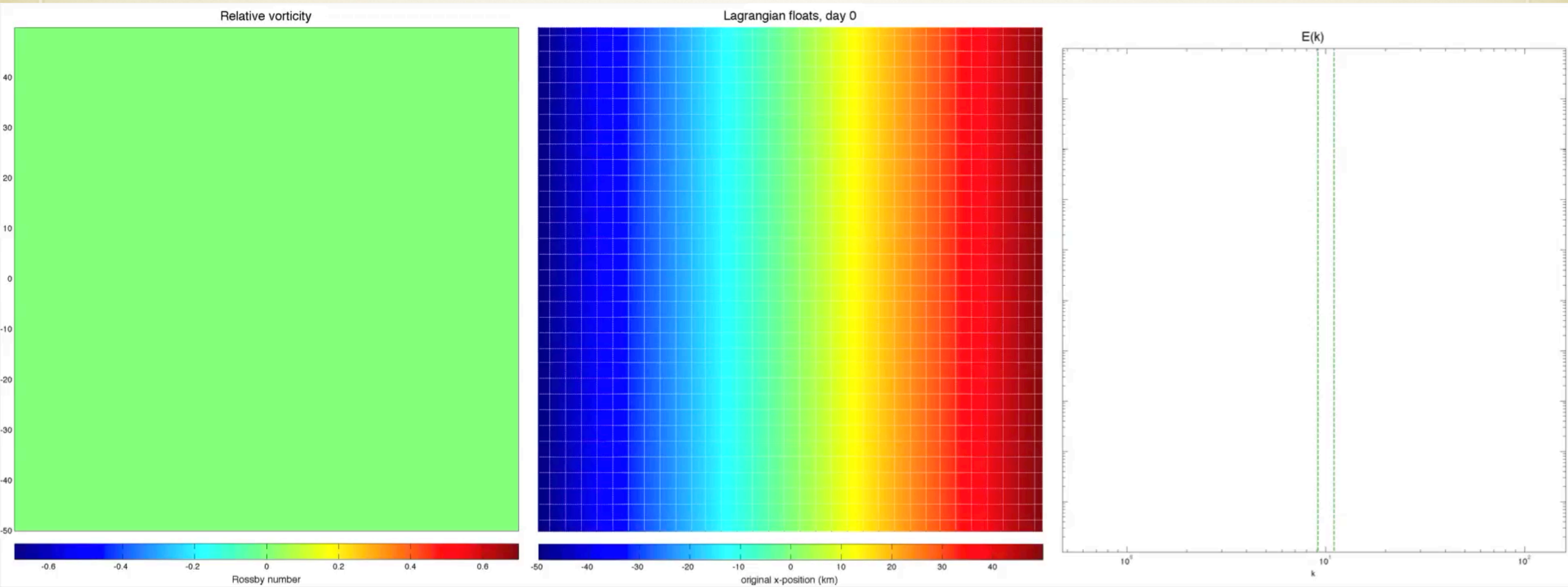
Something damped-slab like reproduces
something that looks like the Complex
Ornstein-Uhlenbeck process.

INGREDIENT 2: TURBULENT BACKGROUND

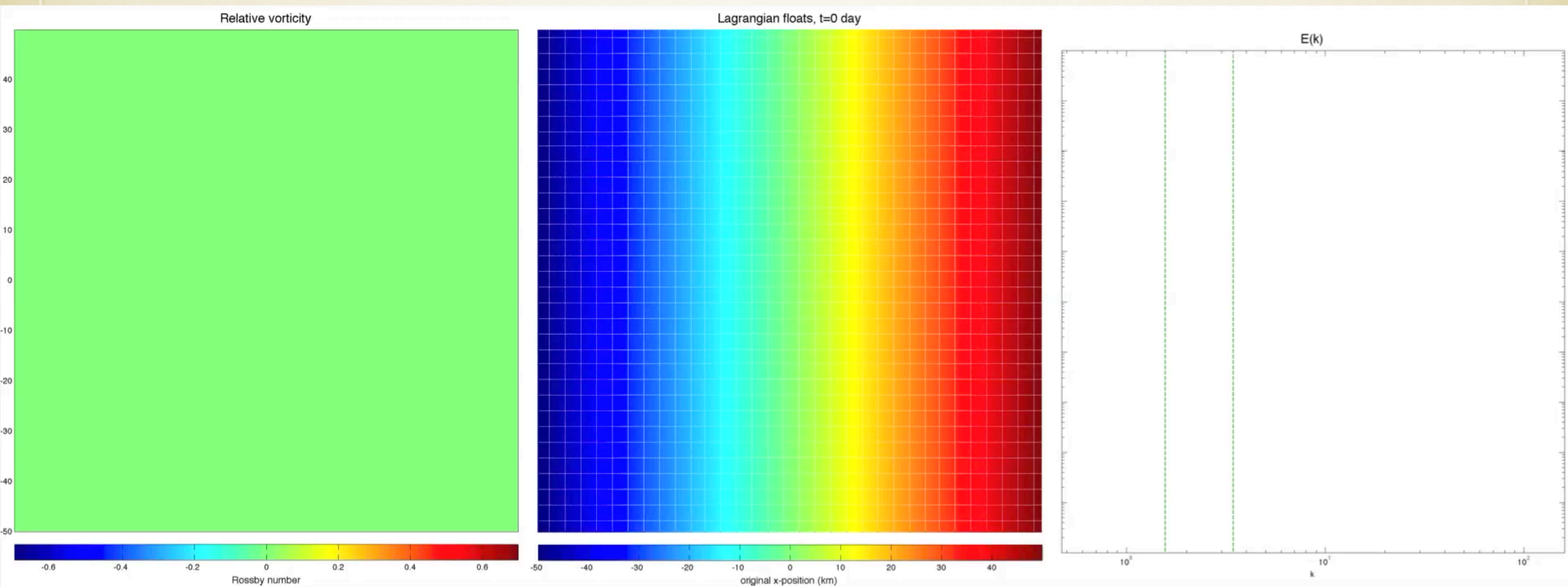


Quasigeostrophic (QG) dynamics reproduces something that looks like the Matérn process.

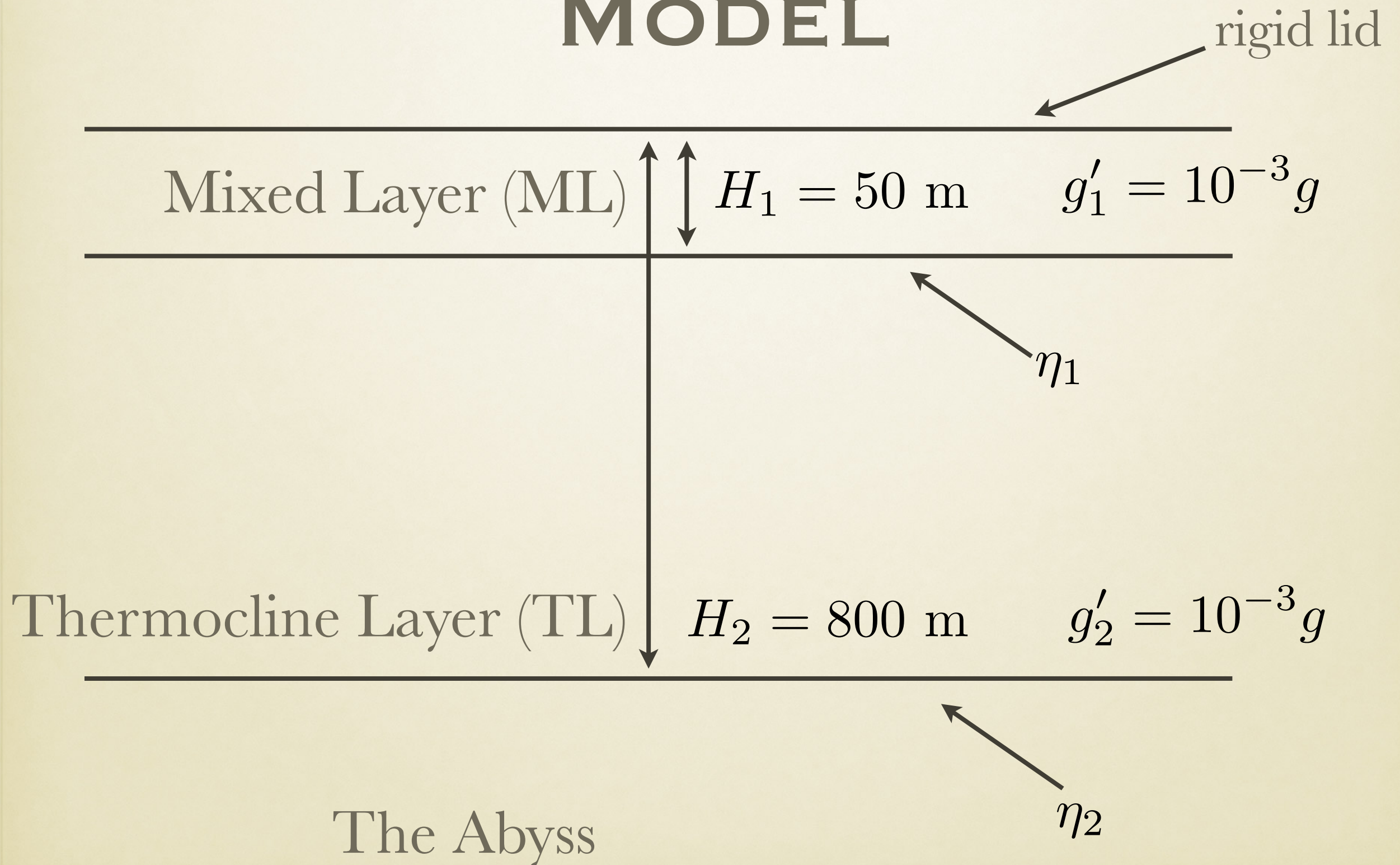
QG TURBULENCE



QG TURBULENCE



PROPOSED DYNAMICAL MODEL



EQUATIONS OF MOTION

Momentum

$$\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 + \mathbf{f} \times \mathbf{u}_1 = \nabla (g'_1 \eta_1 + g'_2 \eta_2) + \frac{\tau}{\rho_1 h_1}$$

$$\frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2 \cdot \nabla) \mathbf{u}_2 + \mathbf{f} \times \mathbf{u}_2 = \nabla g'_2 \eta_2$$

Continuity

$$\frac{\partial h_1}{\partial t} = - \nabla \cdot (h_1 \mathbf{u}_1)$$

$$\frac{\partial h_2}{\partial t} = - \nabla \cdot (h_2 \mathbf{u}_2)$$

2 INTERNAL MODES

- Barotropic Mode (ML & TL motions)

$$\eta_1 \sim \frac{H_1}{H_2} \eta_2 \quad u_1 \sim u_2 \quad c^2 = g(H_1 + H_2)$$

- Baroclinic Mode (Primarily ML motions)

$$\eta_2 \sim -\frac{H_1}{H_2} \eta_1 \quad u_2 \sim -\frac{H_1}{H_2} u_1 \quad c^2 = gH_1$$

➡ Near geostrophic barotropic mode, and near inertial baroclinic mode would be a good model.

EQUATIONS OF MOTION

Momentum

$$\frac{\partial \mathbf{u}_I}{\partial t} + (\mathbf{u}_I \cdot \nabla) \mathbf{u}_G + ((\mathbf{u}_I + \mathbf{u}_G) \cdot \nabla) \mathbf{u}_I + \mathbf{f} \times \mathbf{u}_I = \nabla g'_1 \eta_1 + \frac{\tau}{\rho_1 h_1}$$

$$\frac{\partial \mathbf{u}_G}{\partial t} + (\mathbf{u}_G \cdot \nabla) \mathbf{u}_G + \mathbf{f} \times \mathbf{u}_G = \nabla g'_2 \eta_2$$

Continuity

$$\frac{1}{H_1} \frac{\partial \eta_1}{\partial t} + \frac{1}{H_2} \left(\frac{\partial \eta_1}{\partial t} - \frac{\partial \eta_2}{\partial t} \right) = \nabla \cdot \mathbf{u}_I$$

$$\frac{1}{H_2} \left(\frac{\partial \eta_1}{\partial t} - \frac{\partial \eta_2}{\partial t} \right) = - \nabla \cdot \mathbf{u}_G$$

SCALES

- ‘Inertial’ equation scales

$$U_I, T_I, L_I, N_1 \qquad R_1 \equiv \frac{g'_1 H_1}{f_0^2} = 12 \text{ km}$$

- ‘Geostrophic’ equation scales

$$U_G, T_G, L_G, N_2 \qquad R_2 \equiv \frac{g'_1 H_2}{f_0^2} = 47 \text{ km}$$

- Already fixed scales

$$\frac{H_1}{H_2} \sim O(\epsilon) \qquad \frac{R_1^2}{R_2^2} \sim O(\epsilon)$$

‘EASY’ ASSUMPTIONS

- Primary inertial balance requires,

$$T_I \approx f_0^{-1}$$

- Primary geostrophic balance requires,

$$T_G \approx \frac{L_G}{U_G}$$

- No velocity scale separation,

$$U_G \approx U_I$$

LENGTH SCALE ASSUMPTION

- Pure inertial oscillations have infinite length scale, so we can assume

$$L_G/L_I \approx \epsilon$$

- However, if the length scale of the geostrophic eddies is modulating the inertial oscillations, then

$$L_G/L_I \approx O(1)$$

- but maybe the *effect* is lower order? We start with,

$$L_G/L_I \approx \epsilon$$

THE FINAL ASSUMPTION

- Changes in the mixed layer thickness use time scale

$$T_I$$

- and changes in the second layer thickness use time scale

$$T_G$$

- This is a cheat. I'm giving η_1 two different scales depending on its context. Although, thought of as w , this may make sense.

ALL ARE SCALES FIXED

- These choices of scales fix $N_1/N_2 = O(1)$
 $U_I/U_G = O(1)$
 $T_I/T_G = O(\epsilon)$
 $L_I/L_G = O(1/\epsilon)$
- The relative magnitude of all the terms in the equations are now determined.
- So let's do an asymptotic expansion!

$$u_I = u_I^0 + u_I^1 + u_I^2 + \dots$$

AT $O(1)$

Inertial Equations

$$\partial_t u_I^0 - f v_I^0 = \tau_x$$

$$\partial_t v_I^0 + f u_I^0 = \tau_y$$

$$\partial_t \eta_1^0 = \partial_x u_I^0 + \partial_y v_I^0$$

Geostrophic Equations

$$-f v_G^0 = \partial_x \eta_2^0$$

$$f u_G^0 = \partial_y \eta_2^0$$

$$0 = \partial_x u_G^0 + \partial_y v_G^0$$

THROUGH $O(E)$

Inertial Equations

$$\partial_t u_I + u_I \partial_x u_G + v_I \partial_y u_G - f v_I = \partial_x \eta_1 + \tau_x$$

$$\partial_t v_I + u_I \partial_x v_G + v_I \partial_y v_G + f u_I = \partial_y \eta_1 + \tau_y$$

$$\partial_t \eta_1 = \partial_x u_I + \partial_y v_I$$

Geostrophic Equations

$$-f v_G = \partial_x \eta_2$$

$$f u_G = \partial_y \eta_2$$

$$\nabla_t^2 \eta_2 - \partial_t \eta_2 + J(\eta_2, \nabla^2 \eta_2) = 0$$

UPSHOT

- Geostrophic motions evolve independent of the inertial motions.
- Inertial motions couple to the (relatively) slowly varying geostrophic background.
- These equations meet the original goal.
- Result consistent with Young & Ben Jelloul

FREQUENCY SHIFTING

- Equations of motion for the inertial motions are,

$$u_t + uU_x + vU_y - fv - g\eta_x = 0$$

$$v_t + uV_x + vV_y + fu - g\eta_y = 0$$

$$\eta_t - H_1 u_x - H_1 v_y = 0$$

- But can be rewritten as...

FREQUENCY SHIFTING

Equations of motion

$$u_t + \left[\frac{\delta + \sigma_n}{2} \right] u + \left[\frac{\sigma_s - \zeta}{2} - f \right] v - g\eta_x = 0$$

$$v_t + \left[\frac{\sigma_s + \zeta}{2} + f \right] v + \left[\frac{\delta - \sigma_n}{2} \right] u - g\eta_y = 0$$

$$\eta_t - H_1 u_x - H_1 v_y = 0$$

where

$$\zeta = V_x - U_y \quad \text{vorticity}$$

$$\delta = U_x + V_y \quad \text{divergence}$$

$$\sigma_s = V_x + U_y \quad \text{shear strain}$$

$$\sigma_n = U_x - V_y \quad \text{normal strain}$$

FREQUENCY SHIFTING

- Pure inertial solutions are periodic with frequency

$$\omega = -i\frac{\delta}{2} \pm \sqrt{\left(f + \frac{\zeta}{2}\right)^2 - \frac{\sigma^2}{4}}$$

where

$$\sigma^2 = \sigma_s^2 + \sigma_n^2$$

WHAT'S NEXT

- How do we handle damping in the mixed layer?
- Try the asymptotic analysis using multiple time scales in order to avoid the goofy time scaling.
- Model the coupled the equations and see how close we get to observed parameters with this first guess.
- How does this look compared to drifter from a primitive equation model (e.g., Danioux)?