

# Group Foliation and the Korteweg-de Vries Equation

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## Research Objective

Use the method of group foliation to find solutions to problems in geophysical fluid dynamics.

- ▶ Our focus in geophysical fluid dynamics has been on explaining non-linear extensions to the theory of Rossby waves.
- ▶ Group foliation uses the symmetries of a partial differential equation (PDE) to simplify the equation without eliminating solutions.

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# Introduction

## 1. Satellite observations and Rossby Waves.

- ▶ Rossby waves propagate slowly westward across the ocean basins.
- ▶ Satellite observations suggest many non-linear characteristics to these waves.
- ▶ The Flierl-Petviashvili (FP) equation solutions might share similar characteristics.
- ▶ Korteweg-de Vries (KdV) equation is a one-dimensional approximation to the FP equation.

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## 2. KdV equation, solution, and derivation with group foliation.

- ▶ Group foliation expresses the equation and solution without its symmetries.
- ▶ Two approaches:
  1. Write equation with differential invariants of the symmetry group.
  2. Write as exterior differential system, 'orthogonal' to the symmetries.
- ▶ Reduced equation and solution are simpler and lack the symmetry.
- ▶ Reconstruction of the solution adds the action of symmetries back, we recover KdV solution.

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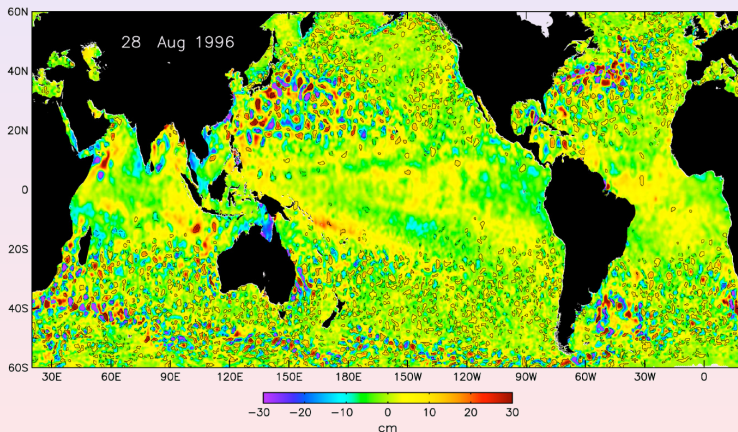
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# Observed Sea Surface Height Anomaly ( $\sim \psi$ )

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- ▶ 18+ week coherent structures are contoured.
- ▶ 50,000 such features in 14 years of data.



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# Rossby Wave Theory

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- ▶  $\psi$  proportional to sea surface height.

## Primary Long Wave Balance

$$\psi_t - \psi_x = 0$$

Solved by any function  $\psi(x, t) = f(x + t)$ .

- ▶ Predicted propagation speed is too slow.
- ▶ No predicted structure.
- ▶ Linear theory also predicts some dispersion; more than actually observed.

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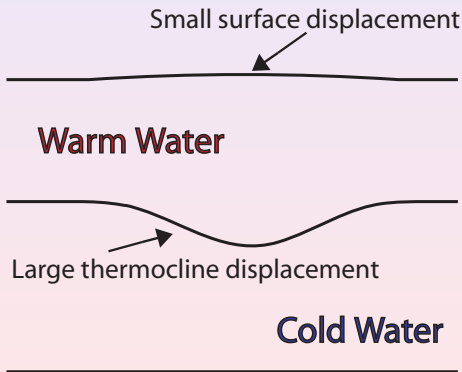
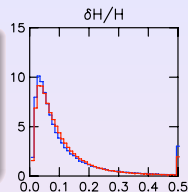
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## One Source of Nonlinearity

Distribution of observed surface displacement to equivalent depth suggests next to leading order terms matter.



## Flierl-Petviashvili Equation

$$\frac{\partial}{\partial t} \left( \nabla^2 \psi - \left( \psi - \frac{1}{2} \psi^2 \right) \right) + \psi_x + J(\psi, \nabla^2 \psi) = 0$$

- ▶ Primary addition is the  $\psi^2$  term.
- ▶ Flierl also looked for eddy like structures, but a shear flow was his primary source of nonlinearity.
- ▶ Petviashvili considered the Great Red Spot on Jupiter.

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## Rossby Waves

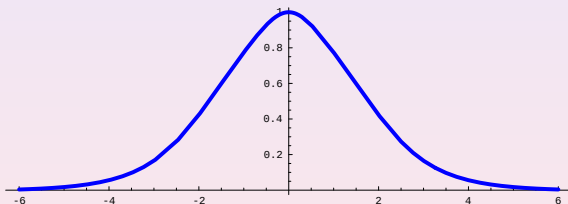
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## Hints of Interesting Solutions

- ▶ One known radial solution, found numerically.
- ▶ Does provide a speedup from linear theory.
- ▶ Positive sea surface height anomaly only (no waves of depression).



# We Want More

## What other solutions can we find?

## Finding Solutions with Group Foliation

## Group Foliation & the KdV Equation

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## Plan of Attack

Find solution to the FP equation with group foliation by first removing its symmetries, and solving the reduced equation.

- ▶ Start with the FP equation: assume a traveling wave solution and  $\frac{d}{dy} = 0$ , you get the KdV equation. Perfect place to start, even has similar symmetries.
- ▶ Advantage that solution is known, therefore helpful to validate new technique.

## Korteweg-de Vries Equation

$$u_{xxxx} - u_x + uu_x = 0$$

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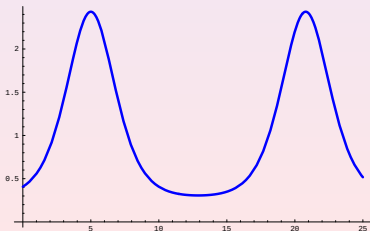
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# Solution of the KdV Equation

- We will refer back to this solution and these constants throughout the foliation.

## Eventual Solution

$$u(x) = 1 + \alpha^2 \left( 1 - k^2 \left( 2 - 3 \operatorname{cn}^2 \left( \alpha \frac{x - x_0}{2}, k^2 \right) \right) \right).$$



Limit solution set to real and bounded. **Three constants:**

1.  $\alpha$  scales the length and amplitude,
2.  $x_0$  translates the origin,
3.  $k$  varies amplitude and the modulus of  $\operatorname{cn}$ .

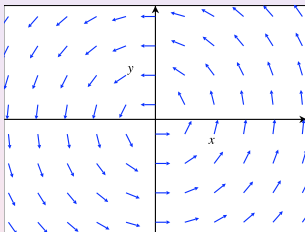
# Symmetries of Partial Differential Equations

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## Definition

A **symmetry of a PDE** takes any solution  $u$  of the PDE and maps it to another solution  $\tilde{u}$  of the PDE.



- ▶ Form a continuous or discrete group.
- ▶ Use the correspondence between a vector field,  $\vec{v}(x, y)$ , and a one-parameter group,  $g(\epsilon; x, y) = \exp[\epsilon \vec{v}]$ .

## Symmetry Condition

Given a differential equation  $\Delta(x, u, u_x, \dots) = 0$ , we want all vector fields that satisfy  $\mathcal{L}_{\text{pr}\vec{v}}\Delta = 0$  when  $\Delta = 0$ .

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There are two point symmetries,

| Infinitesimal Symmetry                            | Finite Symmetry                                            |
|---------------------------------------------------|------------------------------------------------------------|
| $\mathbf{v}_x = \partial_x$                       | $(x, u) \mapsto (x + \epsilon, u)$                         |
| $\mathbf{v}_s = x\partial_x + 2(1 - u)\partial_u$ | $(x, u) \mapsto (e^\epsilon x, 1 - e^{-2\epsilon}(1 - u))$ |

- ▶ Each of these symmetries rescales one of the constants in the solution:  $\mathbf{v}_x$  scales  $x_0$  and  $\mathbf{v}_s$  scales  $\alpha$ .

## Solution

$$u(x) = 1 + \alpha^2 \left( 1 - k^2 \left( 2 - 3 \operatorname{cn}^2 \left( \alpha \frac{x - x_0}{2}, k^2 \right) \right) \right).$$

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## Central Idea

Rewrite the PDE by removing its point symmetries, effectively removing the constants of integration.

- ▶ In our case we will remove the  $\mathbf{v}_x$  scaling of  $x_0$  and  $\mathbf{v}_s$  scaling of  $\alpha$ .
- ▶ Two approaches:
  1. Write equation with differential invariants of the symmetry group.
  2. Write as exterior differential system, 'orthogonal' to the symmetries.

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1. Find the differential invariants, operators of invariant differentiation and the syzygies – these are all properties of the group, found from the group action.
2. Rewrite the PDE using the differential invariants as new coordinates.
3. Solve the reduced equation.
4. Return to original coordinates by performing one more integration with the differential invariants.

## Definition

**Differential invariants** are functions of jet space coordinates ( $x, u, u_x, u_{xx}, \dots$ ) that are invariant under the symmetry operation.

- ▶ Any functional combination of differential invariants, is another differential invariant.

## Example

KdV equation's symmetry group:  $\frac{\psi_x}{(1-\psi)^{3/2}}, \frac{\psi_{xx}}{(1-\psi)^2}, \frac{\psi_{xxx}}{(1-\psi)^{5/2}},$  etc.

# Operators of Invariant Differentiation

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## Definition

The **operators of invariant differentiation** are differential operators that take an  $n$ -th order invariant to an  $(n+1)$ -th order invariant.

- There are a finite number of operators which, combined with the appropriate set of differential invariants, can be used to generate any higher order invariant.

## Example

KdV equation's symmetry group:  $\mathcal{D} = \frac{1}{\sqrt{1-\psi}} D_x$ .

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## Definition

The **syzygies** are functional relationships between the invariants and the differential operators.

- ▶ Differentiating each fundamental invariant with each operator does not necessarily give you an independent set of  $(n+1)$ -th order invariants. The syzygies are the relationships between these invariants.

## Example

KdV equation's symmetry group does not have any syzygies. However, an example from the FP equations symmetry group:  $D_X l_{001} - D_T l_{100} = -l_{100} l_{001}$  where  $l_{nnn}$  are differential invariants.

## Equation on the Quotient Manifold

Write the differential equation in terms of the chosen differential invariants. Given a choice of invariants as the new independent and dependent variables, the differential equation is now of lower order. Combined with the syzygies at that order, we can completely reproduce the original equation.

## Example

KdV equation becomes an Abel equation of the second kind:

$$aa_z = a - \frac{6}{25}z + \frac{2}{5}$$

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# Exterior Differential Systems

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1. Write differential equation as an exterior differential system (EDS), ie, as an ideal of differential forms.
2. Find the reduced EDS by finding the semi-basic forms and choosing a cross-section.
3. Solve – find the integral manifold of the reduced EDS.
4. Reconstruct – rewrite in original coordinates.

# Construct Exterior Differential System

## Example

The KdV equation is described by the ideal,

$$\mathcal{I} = \langle \omega^1 = du - u_x dx, \omega^2 = du_x - u_{xx} dx, \\ \omega^3 = du_{xx} - (1 - u)u_x dx \rangle$$

- ▶ Pullback of the contact forms  $\omega^1$  and  $\omega^2$  ensures derivatives makes sense.
- ▶ Pullback of  $\omega^3$  encodes the KdV equation.

## Example

Let  $\sigma : (x) \rightarrow (x, u, u_x, u_{xx})$  where  
 $\sigma(x) = (x, f(x), f_x(x), f_{xx}(x))$ .

Then  $\sigma^*\omega^1 = 0$  is  $(\frac{df}{dx} - f_x) dx = 0$ . Or just  $\frac{df}{dx} = f_x$ .

# Construct Reduced EDS

The reduced ideal is described by the semi-basic forms.

## Definition

The **semi-basic forms** are the forms in the ideal that vanish with the interior product of the symmetry generators.

## Example

An arbitrary one-form in the ideal is

$$\beta = \beta_1 \omega^1 + \beta_2 \omega^2 + \beta_3 \omega^3.$$

Find the forms that satisfy  $\mathbf{v}_x \lrcorner \beta = 0$  and  $\mathbf{v}_s \lrcorner \beta = 0$ . In our case there is only one.



# Construct Reduced EDS

- ▶ Knowing where we want to go, we choose an appropriate cross-section.
- ▶ Pullback  $\beta$  to the quotient manifold with  $\sigma$ .

## KdV on the Quotient Manifold

$$\sigma^*\beta = \left( a - \frac{6}{25}z + \frac{2}{5} \right) dz - ada.$$

- ▶ This is the same Abel equation as before, written as a differential form.

# Solution on the Quotient Manifold

Any number of techniques can now be used to find the solution to the reduced equation, in our case, an Abel equation of the second kind.

## Solution on the Quotient Manifold

$$a(t) = \frac{C}{5} \frac{(2t+3)}{t^3} \quad (1)$$

$$z(t) = -C \frac{(t+1)}{t^3} + \frac{5}{3}. \quad (2)$$

- ▶ The solution has **one** constant of integration, related to  $k$  in the solution.
- ▶ The other two constants ( $\alpha$  and  $x_0$ ) will be recovered by applying the group action to the above solution.

- ▶ Apply the general group action to reconstruct the solution in the full space,  $s = \mu(\gamma, \sigma \circ \bar{s})$ .

## Solution Reconstruction

$$s(t) = \mu \left( \xi, 1 - \eta, \eta^{3/2} \left( \frac{z}{5} \right)^{1/2}, \frac{\eta^2}{2} \left( a - \frac{3}{5} z \right) \right)$$

- ▶  $a$  and  $z$  is the solution on the quotient manifold, known functions of  $t$ .
- ▶  $\xi$  and  $\eta$  are arbitrary group parameters, unknown functions of  $t$ .
- ▶ The solution is just  $(x(t), u(t)) = (\xi(t), 1 - \eta(t))$ .

## Reconstruction

- Pullback of contact forms provides final integrability conditions.

Require  $\sigma^* (du_x - u_{xx} dx) = 0$

We find

$$\eta = \alpha^2 t$$

where  $\alpha^2$  is the constant of integration.

Require  $\sigma^* (du - u_x dx) = 0$

We find

$$\xi = -x_0 - \frac{\sqrt{3}}{\alpha} \int \frac{dt}{\sqrt{t^3 - C(t+1)}}$$

where  $x_0$  is the constant of integration.

- ▶ The final two integrations added the group parameters back into the equation.

## Solution

$$u(x) = 1 + \alpha^2 \left( 1 - k^2 \left( 2 - 3 \operatorname{cn}^2 \left( \alpha \frac{x - x_0}{2}, k^2 \right) \right) \right).$$

# Conclusions

- ▶ Successfully reconstructed KdV solution using group foliation (but it wasn't easy).
- ▶ Now we are proceeding with the more challenging FP equation.
  - ▶ Foliated system similar to KdV Abel equation, but higher order (system of Abel equations).
- ▶ The technique could prove valuable for equations with larger symmetry groups, like the planetary geostrophic equations with an infinite dimensional group.