

Near-Geostrophic Approximations to the Spherical Shallow-Water Equations

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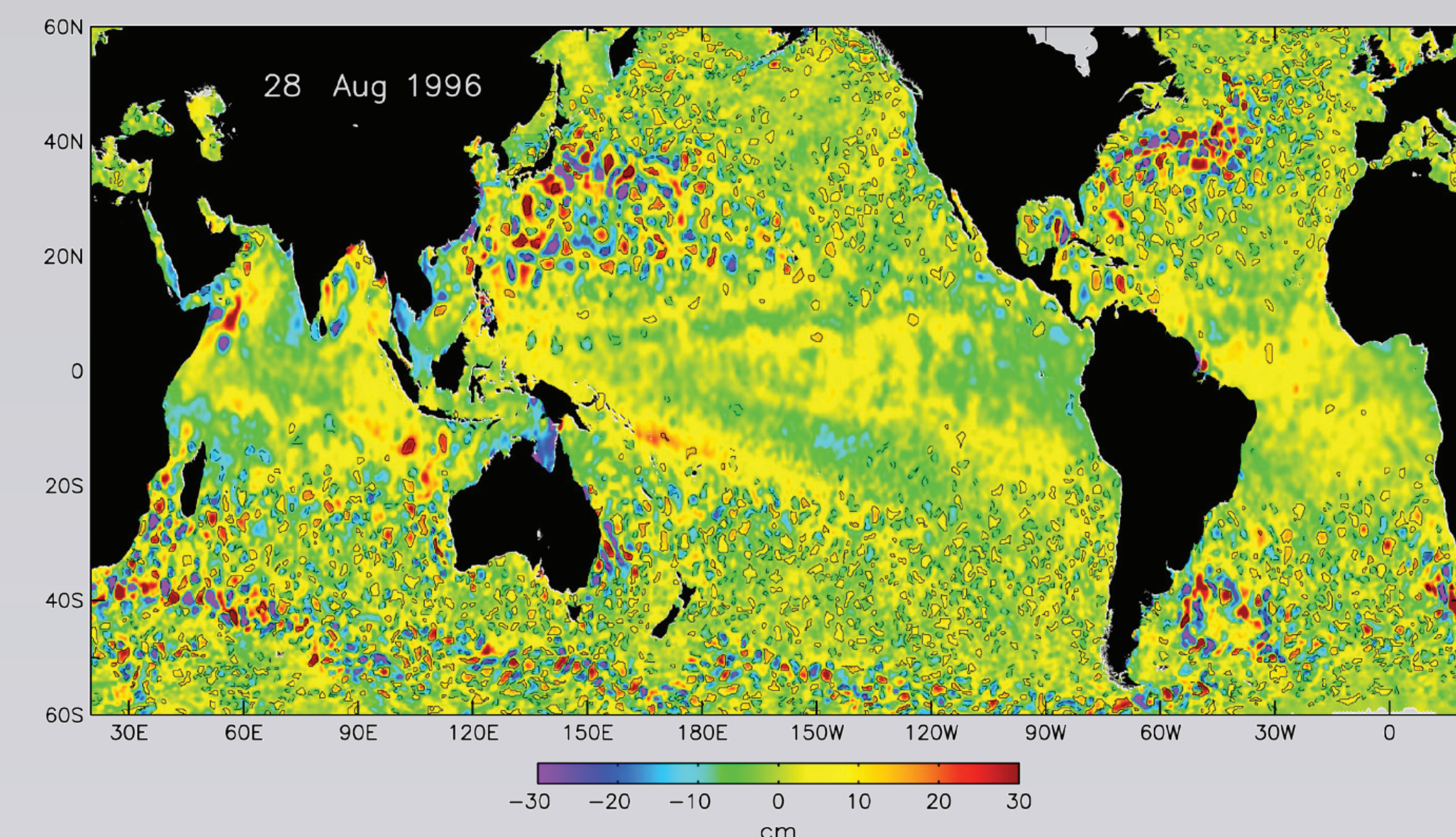
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I. Observational Motivation

Altimeter-tracked mesoscale eddies observed to propagate westward across the ocean basins lack an adequate theoretical explanation. Observations show:

- The eddies propagate slightly faster than long wave phase speed of first baroclinic mode Rossby waves, c .
- Fluid velocities, U , are commonly 5x the translation speed.
- The ratio of the eddy height to the first baroclinic mode equivalent depth, ϵ_H , is commonly 0.1 or more.
- Typical length scales of the eddies exceed 100 km.

These observations all push the boundaries of quasi-geostrophic theory.



Sea surface height anomaly. 18+ week coherent structures are contoured. 50,000 such features in 14 years of data. Image courtesy of D. Chelton & M. Schlax.

II. Central Questions

- We know a lot about the scales that the eddies are evolving at; can we use this information to find an appropriate theory?
- Conversely, given a theory like quasi-geostrophy, can we determine the range of scales where the theory is valid?

III. Non-dimensionalization

- Remove the units of the equation and find the non-dimensional parameters. There are five that naturally arise in our case, six if you count latitude.

$$\epsilon_T = \frac{1}{2\omega T}, \quad \epsilon_R = \frac{U}{2\omega L}, \quad \epsilon_\beta = \frac{U}{2\omega R}, \quad \epsilon_H = \frac{N_0}{D}, \quad G = \frac{gN_0}{2\omega UL}$$

- Length (L), fluid speed (U), surface height (N_0), layer depth (D), earth rotation rate (ω), earth radius (R), gravity acceleration (g).

- Non-dimensionalized shallow water equations,

$$\begin{aligned} \bar{v} &= \frac{G}{\sin \theta \cos \theta} \frac{\partial \eta}{\partial \bar{x}} + \epsilon_T \cdot \frac{1}{\sin \theta} \frac{\partial \bar{u}}{\partial \bar{t}} + \epsilon_R \cdot \frac{1}{\sin \theta} \left[\frac{\bar{u}}{\cos \theta} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \right] - \epsilon_\beta \cdot \frac{\bar{u} \bar{v}}{\cos \theta}, \\ \bar{u} &= -\frac{G}{\sin \theta} \frac{\partial \eta}{\partial \bar{y}} - \epsilon_T \cdot \frac{1}{\sin \theta} \frac{\partial \bar{v}}{\partial \bar{t}} - \epsilon_R \cdot \frac{1}{\sin \theta} \left[\frac{\bar{u}}{\cos \theta} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right] - \epsilon_\beta \cdot \frac{\bar{u}^2}{\cos \theta}, \\ 0 &= \frac{\epsilon_H \epsilon_T}{\epsilon_R} \eta_t + \epsilon_H \frac{\bar{u}}{\cos \theta} \frac{\partial \eta}{\partial \bar{x}} + \epsilon_H \bar{v} \frac{\partial \eta}{\partial \bar{y}} + \frac{1 + \epsilon_H \eta}{\cos \theta} \left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{y}} (\bar{v} \cos \theta) \right). \end{aligned}$$

IV. The Asymptotic Approach

- Expand the three variables, u , v , η , in terms of ϵ .

$$v = v^0 + \epsilon v^1 + \epsilon^2 v^2 + O(\epsilon^3)$$

- One must fix the ratio of the non-dimensional parameters to one another. This is a mathematical requirement of asymptotics, not a physical requirement.

- Given a solution with different parameter ratios, how does one know if the equation is still valid in that regime?

- Warn, et. al. showed that expanding η generally leads to instabilities; instead it should be left unexpanded, as a ‘slaving variable.’

Warn, T, O. Bokhove, T.G. Shepherd, and G.K. Vallis, Rossby number expansions, slaving, and balance dynamics. Quart. J. Roy. Meteor. Soc., 121, 723-739 (1995).

V. A New Approach

Do not fix the parameters, do not expand the variables, simply substitute to find the next order.

As an example take these simplified equations,

$$\begin{aligned} v &= G\eta_x + \epsilon u_t \\ u &= -G\eta_y - \epsilon v_t \end{aligned}$$

and substitute them into each other,

$$\begin{aligned} v &= G\eta_x - \epsilon G\eta_{yt} - \epsilon^2 v_{tt} \\ u &= -G\eta_y - \epsilon G\eta_{xt} - \epsilon^2 u_{tt}. \end{aligned}$$

- Provided $G \gg \epsilon_T \epsilon_R \epsilon_\beta$, u and v can be written to any order accuracy, simply truncate at ϵ^n .

- In our case, the potential vorticity equation (PVE) is entirely in terms of η through two orders of $\epsilon_T \epsilon_R \epsilon_\beta$, with the truncation error at three orders.

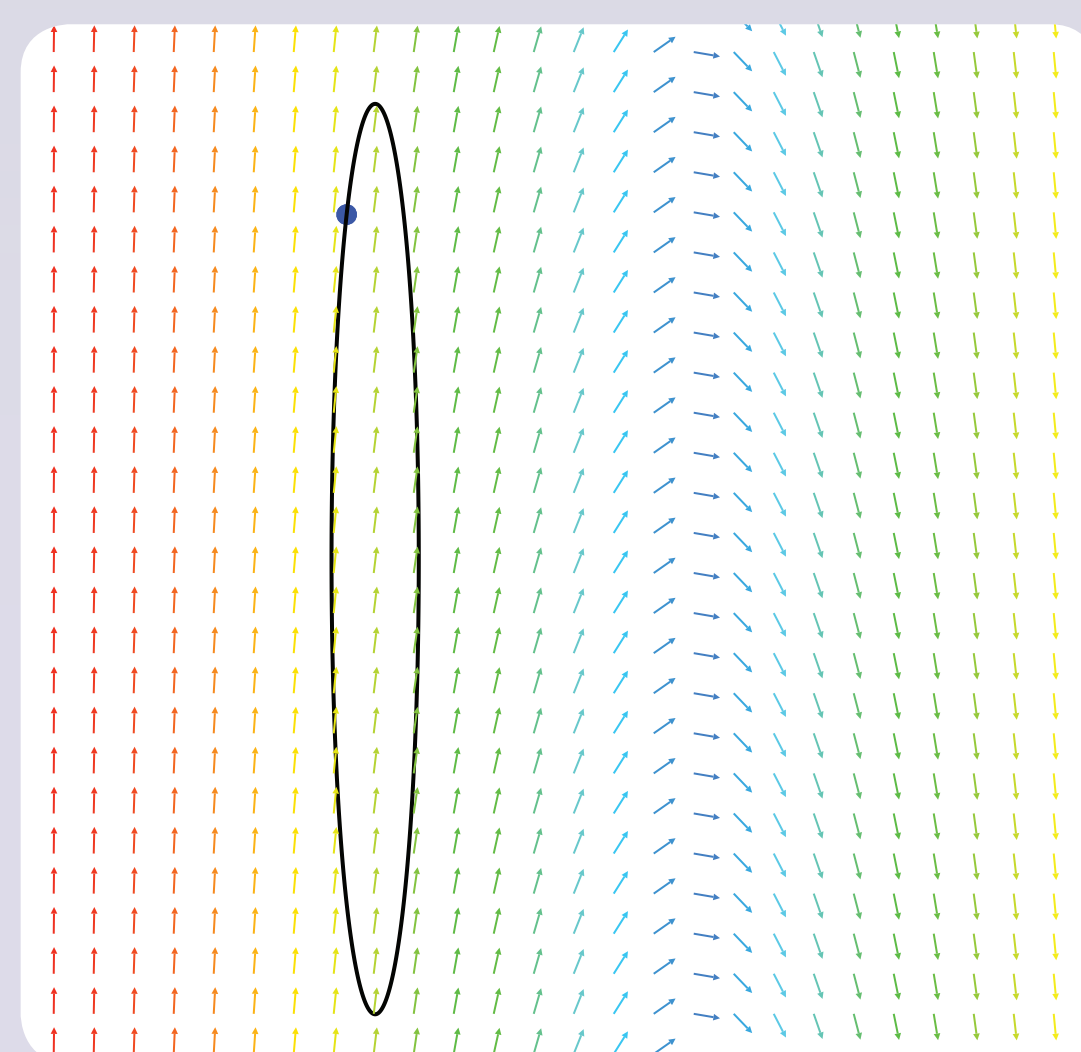
- Use the non-dimensional parameters to estimate the size of the terms, and the truncation error.

- Trigonometric terms are also expanded to determine the spherical geometry truncation error.

- PVE contains a finite number of terms, each with non-dimensional coefficient. The equation remains exact until we truncate to remove the last terms with u and v .

- Equivalent to slaving variable approach (expanding u and v only) once the scales are fixed.

Rossby Wave Particle Path

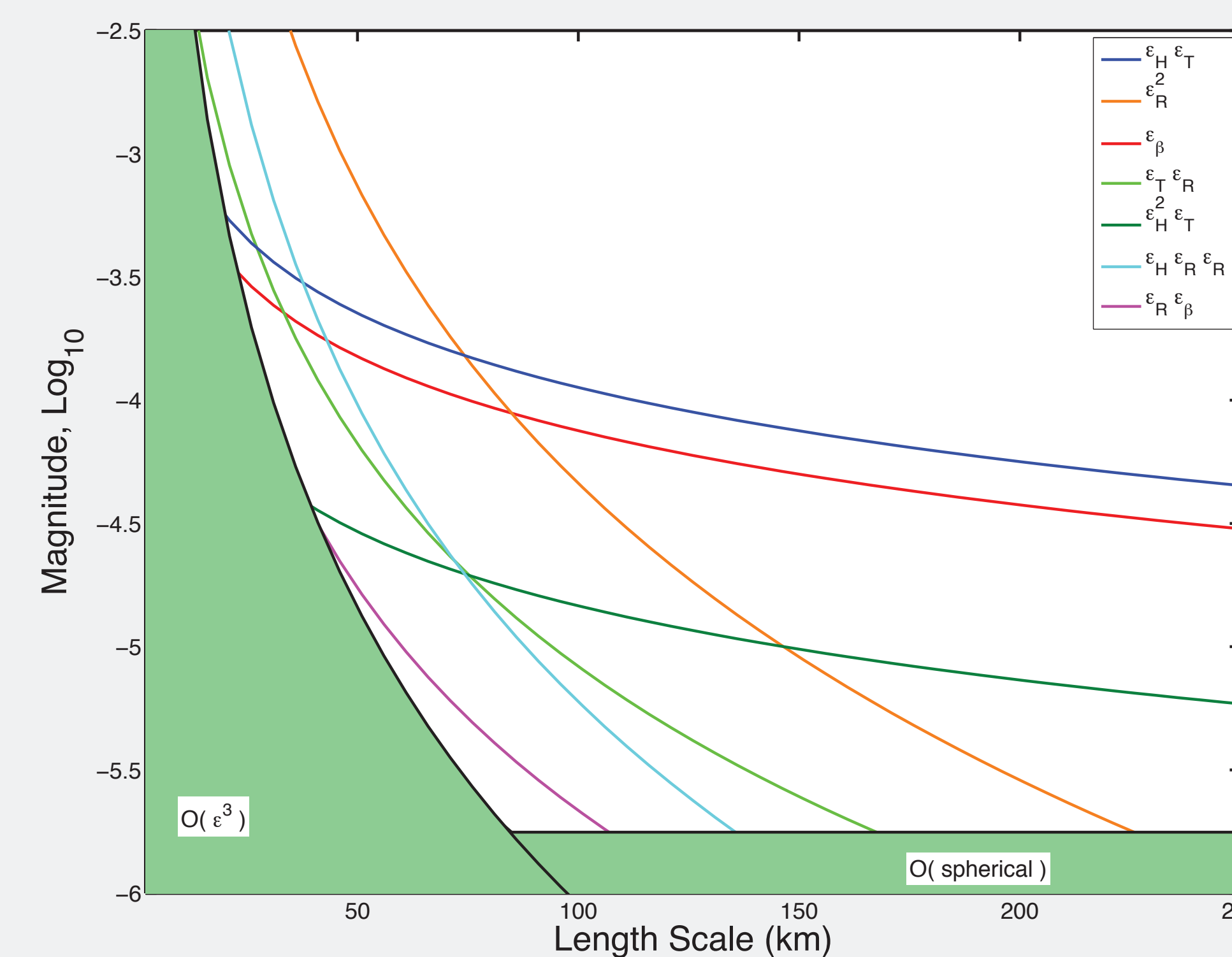


The vector field represents the velocity field of a long wave Rossby wave propagating westward and the ellipse is the path of a fluid particle. The asymptotic approach is only able to determine the north-south component of the fluid's motion.

VI. A Generalized Geostrophic Potential Vorticity Equation

The magnitude of the terms are shown versus the length scale. The surface height is fixed at 10 cm, the ratio of the length scale to the time scale at 1.8 cm/s, the latitude at 45, and the fluid velocity at 10 cm/s.

Magnitude of Terms

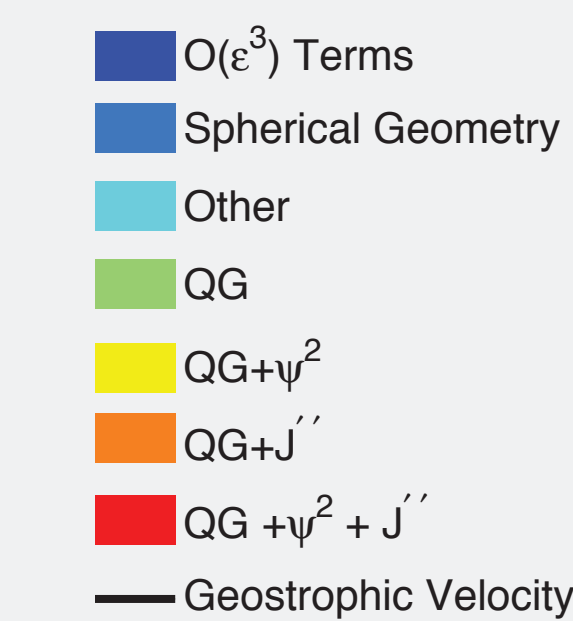


$$\begin{aligned} 0 &= \epsilon_\beta G \cdot \frac{\cos \theta}{\sin \theta} \frac{\eta_x}{\cos \theta} && \leftarrow \text{QG term} \\ &+ \epsilon_T \epsilon_R G \cdot \frac{1}{\sin \theta} \nabla^2 \eta_t && \leftarrow \text{QG term} \\ &- \epsilon_T \epsilon_\beta G \cdot \frac{1 + \cos^2 \theta}{\sin^2 \theta \cos \theta} \eta_{ty} && \\ &+ \epsilon_R^2 G^2 \cdot \frac{1}{\sin^2 \theta \cos \theta} \left[\frac{\eta_y \eta_{xxx}}{\cos^2 \theta} + \eta_x \eta_{yyy} - \eta_y \eta_{xyy} + \frac{\eta_x \eta_{xxy}}{\cos^2 \theta} \right] && \leftarrow \text{QG term} \\ &+ \epsilon_R \epsilon_\beta G^2 \cdot \frac{\cos \theta}{\sin^3 \theta} \left[-3 \frac{\eta_x \eta_{yy}}{\cos \theta} + \frac{1 + \cos^2 \theta}{\cos^2 \theta} \frac{\eta_y \eta_{xy}}{\cos \theta} + \frac{2 - 3 \cos^2 \theta}{\cos^2 \theta} \frac{\eta_x \eta_{xx}}{\cos^3 \theta} \right] \\ &- \epsilon_R \epsilon_\beta G \cdot \frac{\eta_{yy}}{\cos \theta} && \\ &+ \epsilon_\beta^2 G^2 \cdot \frac{\cos^4 \theta + 3 \cos^2 \theta - 1}{\sin^4 \theta \cos^2 \theta} \frac{\eta_x \eta_y}{\cos \theta} && \\ &+ \epsilon_\beta^2 G \cdot \frac{1}{\sin \theta} \eta_y && \leftarrow \text{QG term} \\ &- \epsilon_H \epsilon_T \cdot \frac{1}{\sin \theta} \eta_t && \leftarrow \text{QG extension, } \eta^2 \\ &+ \epsilon_H \epsilon_T \epsilon_R G \cdot \frac{1}{\sin \theta} \left[\frac{\partial}{\partial t} \left(\frac{\eta_x^2}{\cos^2 \theta} + \eta_y^2 \right) - \eta_t \cdot \nabla^2 \eta \right] \\ &+ \epsilon_H \epsilon_T \epsilon_\beta G \cdot \frac{\eta_y \eta_t}{\sin^2 \theta \cos \theta} \\ &- \epsilon_H \epsilon_R^2 G^2 \cdot \frac{1}{\sin^2 \theta \cos \theta} \left[\eta_x \eta_y \left(\frac{\eta_{xx}}{\cos^2 \theta} - \eta_{yy} \right) + \eta_{xy} \left(\eta_y^2 - \frac{\eta_x^2}{\cos^2 \theta} \right) \right] && \leftarrow \text{QG extension, } J'' \\ &- \epsilon_H \epsilon_R \epsilon_\beta G^2 \cdot \frac{\cos \theta}{\sin^3 \theta \cos \theta} \left[\frac{\cos 2\theta}{\cos^2 \theta} \frac{\eta_x^2}{\cos^2 \theta} + \eta_y^2 \right] + O(\epsilon^3) \end{aligned}$$

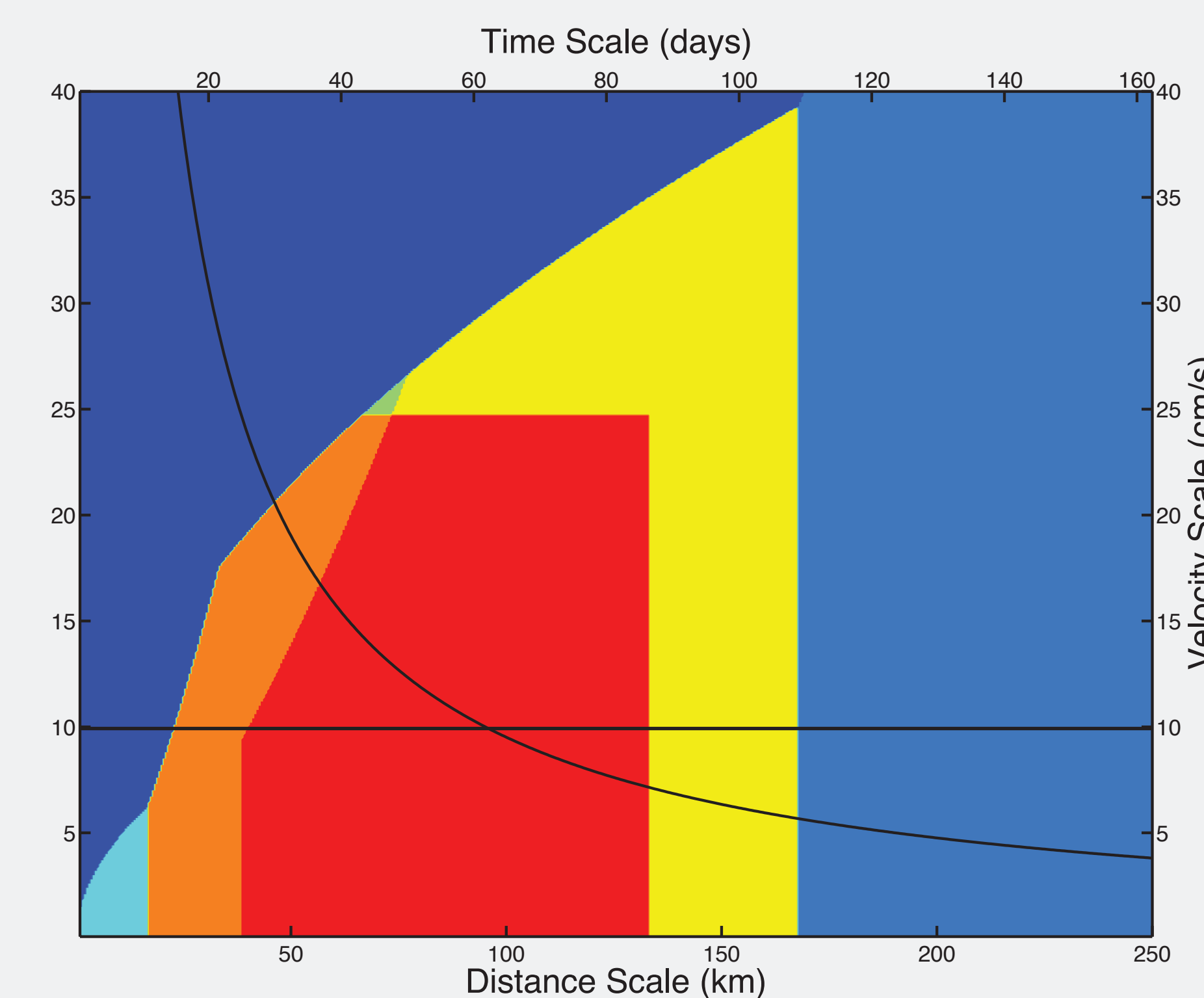
VII. Valid Areas of Parameter Space for Candidate Theories

Each colored region shows the area in parameter space where a given theory is valid (all terms have the greatest magnitude). The blue regions require additional terms from either the next order expansion $O(\epsilon^3)$, or from expansion of trigonometric terms $O(\text{spherical})$.

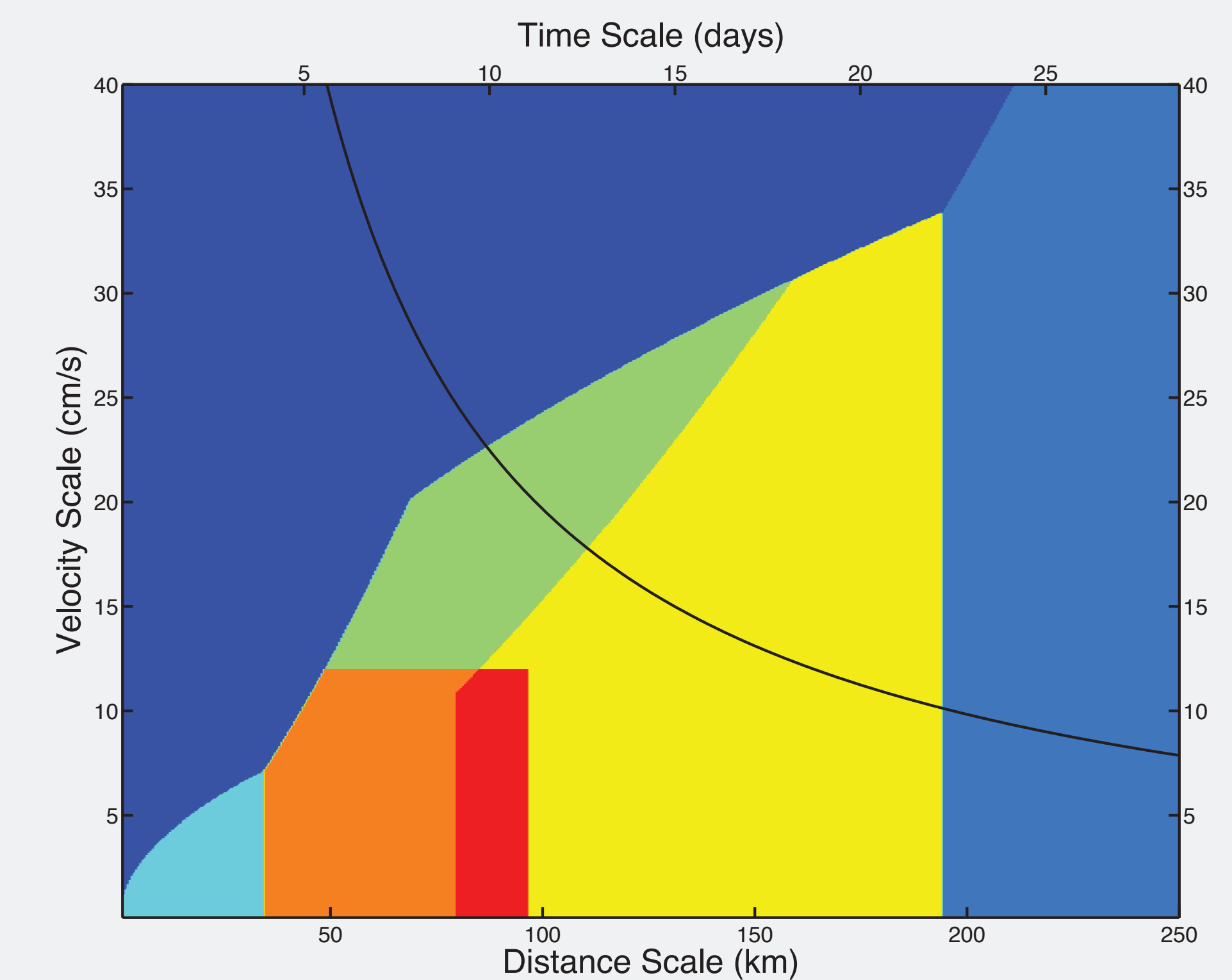
- Surface height at 10 cm ($\epsilon_H = 0.13$)
- Ratio of length scale to time scale fixed at 1.5 x long Rossby wave phase speed.



Latitude 45, L/T = 1.8 cm/s



Latitude 20, L/T = 10.2 cm/s



VIII. Conclusions

From these plots we can see that the region of interest is dominated by extensions to quasi-geostrophic theory.

- The addition of the η^2 term to QG is known as the Flierl-Petviashvili equation. The term reflects a local nonlinear sea surface height change, the next order term to η_t .
- The J'' term may also be important in the description of the eddies. This is an ageostrophic term describing the advection of fluid associated with the height nonlinearity, essentially the next order term to the Jacobian.