

Group Foliation and the FP Equation

Jeffrey J. Early

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Physical Interest

Rossby Waves

FP Equation

Solutions

Mathematical Methods

Symmetries

Symmetry reduction

Group Foliation

Results

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- ▶ Slowly propagating westward.
- ▶ Vorticity restoring force.
- ▶ Transfers energy to the western boundary currents.
- ▶ SSH signature detectable with satellite.

Rossby Wave Observations

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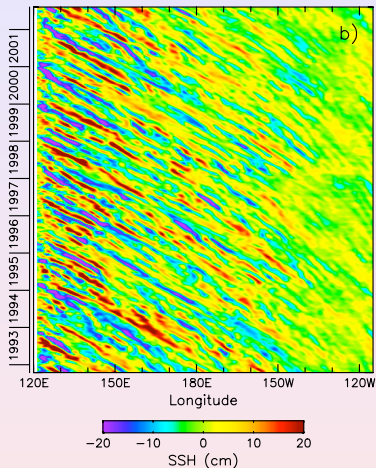


Figure: Time-longitude plot of filtered SSH along 24N, from Dudley Chelton.

Problems with classical theory

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- ▶ Propagation speeds don't jive.
- ▶ Dispersion relation is non dispersive, even at small wavelengths.
- ▶ Upper layer thickness is too big for linear assumptions.
- ▶ Others....

Problems with classical theory

- ▶ Propagation speeds don't jive.
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- ▶ Others....

Can we address at least one of these problems? Focus on nonlinear thickening.

Flierl-Petviashvili Equation (non-dimensionalized)

$$\nabla^2 \psi - \psi - \psi^2 = 0$$

- ▶ Derived by Glenn Flierl in 1979 studying oceanic baroclinic traveling wave solutions.
- ▶ Also derived by Petviashvili in 1981 studying the Great Red Spot of Jupiter and solitons in plasma.

Derivation

- ▶ Start with potential vorticity conservation,

$$\frac{d}{dt} \frac{\zeta + f}{H} = 0. \quad (1)$$

- ▶ Assume nearly geostrophic motions and define a stream function, ψ .
- ▶ Use a β -plane approximation.
- ▶ Take the length scale to be longer than the Rossby radius of deformation.

Petviashvili Equation

$$\left[\nabla^2 \psi - F_0 \left(\psi - \frac{1}{2} \psi^2 \right) \right]_t + \beta \psi_x = 0.$$

- If you dump the y component and look for traveling wave solutions, the KdV soliton solution emerges,

$$\psi(x, t) = \frac{12}{F_0 l^2} \operatorname{sech}^2 \left(\frac{x - ct}{l} \right) \quad (2)$$

where $c = c_0 \left(1 - \frac{4}{F_0 l^2} \right)^{-1}$.

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- ▶ Upshot: the wave is solitary and does provide a speed up. However, it's one-dimensional!

Solutions

- ▶ Flierl found a radial solitary wave solution numerically in 1979.
- ▶ Collisions of the solitary waves were studied numerically more recently by Boyd and Tan.

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More Analytical Solutions?

Results suggest the FP equation does encode interesting physics, so what else can we extract?

Symmetry of a Differential Equation

A symmetry of a differential equations takes a solution u of the differential equation and maps it to another solution $\tilde{u}$ of the differential equation.

- ▶ The symmetries of a differential equation form a group, G .
- ▶ The symmetries may be continuous or discrete.

Finding Symmetries

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- ▶ The problem is straightforward and algorithmic.
- ▶ Computations can become intractable to do by hand, but symbolic manipulators (like Maple) do a great job.

What you end up with...

A collection of vector field $(\vec{v}_1, \vec{v}_2, \dots)$ which correspond to your transformations.

Using the Symmetries: Generalized Solutions

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- ▶ If you have a solution u to the differential equation, you can apply all of the transformations to it and you still have a solution (by definition).
- ▶ Example: If $u(x, t)$ is a solution to the heat equation, $u_t = u_{xx}$, then so is $f(x, t) = e^{-cx+c^2t}u(x - 2ct, t)$.

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- ▶ Example: If $u(x, t)$ is a solution to the heat equation, $u_t = u_{xx}$, then so is $f(x, t) = e^{-cx+c^2t}u(x - 2ct, t)$.
- ▶ But okay, that's not horribly useful...

Using the Symmetries: Symmetry Reduction

- ▶ Now you postulate there exist solutions completely invariant under one of the symmetries. Meaning that under the transformation $u \mapsto \tilde{u}$, you assume $\tilde{u} = u$. This guarantees that the equation will be simplified.
- ▶ People use this all the time without knowing it: looking for traveling wave solutions.
- ▶ Example: In the heat equation $u_t = u_{xx}$ you might look for solutions of the form $u(x, t) = v(x - ct)$, or $u(x, t) = t^a v(x/\sqrt{t})$.
- ▶ BUT, in exchange for simplifying the equation, we eliminated solutions.

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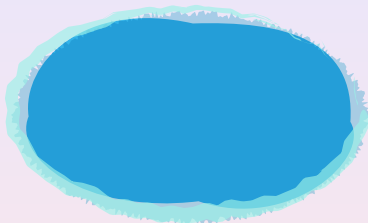


Figure: This blob represents the space of all functions – where solutions to our differential equation live.

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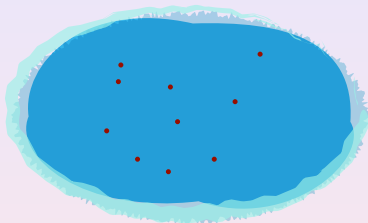


Figure: Points in the blob represent functions. So solutions to a given PDE will be a number of different points – shown in red.

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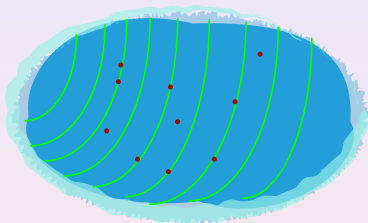


Figure: If the PDE has a symmetry group, solutions will be continuously connected to other solutions, along a green line.

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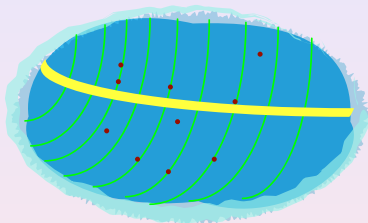


Figure: So rather than consider the whole space, we can just consider a cross-section of the green lines, shown in yellow.

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- ▶ First you need to write down a set of equations describing the orbits. This is the Automorphic System, because it describes how you move one solutions to another.
- ▶ Second you need to write down your equation in these new coordinates. This is known as the Resolving System.
- ▶ Importantly, we have lost NO solution using this technique and have in some sense 'simplified' the solution space. Although in practice it's not quite so simple.

- ▶ Having successfully applied the group foliation method to the FP equation, I vastly simplified this equation:

$$\left[\nabla^2 \psi - F_0 \left(\psi - \frac{1}{2} \psi^2 \right) \right]_t + \beta \psi_x = 0.$$

and turned it into...

$$\mathcal{D}_1 B - \mathcal{D}_2 A = 2(Bq - Ar)$$

$$\mathcal{D}_1 G - \mathcal{D}_2 C = Gq - Cr$$

$$\mathcal{D}_1 G - \mathcal{D}_3 B = Gq - 2Bs$$

$$\mathcal{D}_2 C - \mathcal{D}_3 B = Cr - 2Bs$$

$$\mathcal{D}_2 H - \mathcal{D}_3 G = -Gs$$

The FP equation itself is

$$\mathcal{D}_1 C - Cq + \mathcal{D}_2 G - Gr - F_0 s + \beta q = 0.$$

The three operators take the form,

$$\mathcal{D}_1 = \left(A + \frac{3}{2}q^2 \right) \partial_q + \left(B + \frac{3}{2}qr \right) \partial_r + \left(C + \frac{1}{2}qs \right) \partial_s,$$

$$\mathcal{D}_2 = \left(B + \frac{3}{2}qr \right) \partial_q + \left(E + \frac{3}{2}r^2 \right) \partial_r + \left(G + \frac{1}{2}rs \right) \partial_s,$$

$$\mathcal{D}_3 = \left(C + \frac{3}{2}qs \right) \partial_q + \left(G + \frac{3}{2}rs \right) \partial_r + \left(H + \frac{1}{2}s^2 \right) \partial_s.$$

Going Forward

In the short term,

- ▶ Confirm my derivation is correct.
- ▶ Choose a different cross-section to make the equations simpler.
- ▶ Look for a solution.

In the long term,

- ▶ Consider other forms of the equation (incl. traveling wave).
- ▶ Used generalized symmetry techniques.
- ▶ Consider the planetary geostrophic equations.